Journal of Symbolic Computation 48 (2013) 64–100

Contents lists available at SciVerse ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

# On symplectically rigid local systems of rank four and Calabi–Yau operators

Michael Bogner [1], Stefan Reiter [1]

*Johannes Gutenberg-Universität, Institut für Mathematik, Staudinger Weg 9, 55128 Mainz, Germany*

ARTICLE INFO

*Article history:*
Received 17 June 2011
Accepted 16 November 2011
Available online 9 May 2012

*Keywords:*
Geometric origin
Rigidity
Calabi–Yau operators
Hadamard product
Convolution

ABSTRACT

We classify all $\mathrm{Sp}_4(\mathbb{C})$-rigid, quasi-unipotent local systems and show that all of them have geometric origin. Furthermore, we investigate which of those having a maximal unipotent element are induced by fourth order Calabi–Yau operators. Via this approach, we reconstruct all known Calabi–Yau operators inducing an $\mathrm{Sp}_4(\mathbb{C})$-rigid monodromy tuple and obtain closed formulae for special solutions of them.



## 1. Introduction

Differential operators of *geometric origin* describe periods of families of complex algebraic varieties and have been studied quite extensively during the last fifty years. A special class of such operators are *fourth order differential Calabi–Yau operators* which are related to families of Calabi–Yau threefolds having a large complex structure limit and $h^{2,1} = 1$. A conjectural characterization of those operators from a purely differential algebraic point of view, together with a list of most of the known examples is stated in Almkvist et al. (2010). In particular, they are irreducible, self-dual, Fuchsian, have only zero as exponent at $z = 0$, i.e. the local monodromy at $z = 0$ is maximally unipotent, and satisfy further integrality conditions, see Definition 6.5. The majority of those operators is not constructed from a geometric situation, as only very few examples of this type are known at the moment. Thus it is natural to ask, which of the operators really are of geometric origin and what would be a geometric realization.

It is quite challenging to decide whether a given differential operator has geometric origin or not. Since differential operators of geometric origin are known to have quasi-unipotent local monodromy,

*E-mail addresses:* mbogner@mathematik.uni-mainz.de (M. Bogner), reiters@uni-mainz.de (S. Reiter).

[1] Both authors were partially supported by the SFB Transregio 45.

0747-7171/$ – see front matter 
http://dx.doi.org/10.1016/j.jsc.2011.11.007

the first order ones are exactly those, which have an algebraic solution, see e.g. Simpson (1990). Furthermore, as observed by André (1989, Chapter II), the class of geometric differential operators is preserved by a multitude of constructions as taking subquotients, direct sums, tensor products and Hadamard products. We call an operator which can be obtained in this way *geometrically constructible*. An appropriate method to check whether an operator is geometrically constructible or not is provided by the following investigation of local solutions.

Given a differential operator $L$ of degree $n$ with coefficients in $\mathbb{C}(z)$ and singular locus $S$, a classical theorem due to Cauchy states that for each $x \in \mathbb{P}^1 \setminus S$ we find a basis $F = \{f_1, \dots, f_n\}$ of the $n$-dimensional $\mathbb{C}$-vector space $\mathrm{Sol}(L)_x = \{L(f) = 0 \mid f \text{ is holomorphic in some disc around } x\}$. If we choose a closed path $\gamma$ starting at $x$, analytic continuation of $F$ along $\gamma$ yields a different basis $\tilde{F}$ of $\mathrm{Sol}(L)_x$. The change from $F$ to $\tilde{F}$ only depends on the homotopy class of $\gamma$. The translation of Cauchy's theorem into 20-th century language thus states the following: the operator $L$ induces a local system $\mathbb{L}$ of rank $n$ on $\mathbb{P}^1 \setminus S$ via

$$\mathbb{L}(U) := \bigl\{ f \in \mathcal{O}_{\mathbb{P}^1 \setminus S}(U) \bigm| L(f) = 0 \bigr\}.$$

Furthermore, with respect to an arbitrary base point $x_0 \in \mathbb{P}^1 \setminus S$ this local system naturally induces a representation

$$\rho_{\mathbb{L}} : \pi_1\bigl(\mathbb{P}^1 \setminus S, x_0\bigr) \to \mathrm{GL}(\mathbb{L}_{x_0})$$

of $\pi_1(\mathbb{P}^1 \setminus S, x_0)$, the so-called *monodromy representation*. Its image is called the *monodromy group* associated to $L$. We may choose a set of generators $(\gamma_s)_{s \in S} \subset \pi_1(\mathbb{P}^1 \setminus S, x_0)$, whose elements are just simple loops $\gamma_s$ around each $s \in S$. As $S$ is finite, it can be equipped with an ordering $I$ such that

$$\prod_{i \in I} \gamma_{s_i} = 1 \in \pi_1\bigl(\mathbb{P}^1 \setminus S, x_0\bigr)$$

holds. Thus the monodromy group is completely determined by the tuple

$$(T_{s_i})_{i \in I} := \bigl(\rho_{\mathbb{L}}(\gamma_{s_i})\bigr)_{i \in I}$$

of linear maps, which fulfill $\prod_{i \in I} T_{s_i} = \mathrm{id}_{\mathbb{L}_{x_0}}$. This tuple $(T_s)_{s \in S}$ is called the *monodromy tuple* associated to $L$ and represents the effect of analytic continuation of holomorphic solutions near $x_0$ around each singularity of $L$. We call a monodromy tuple to be of *geometric origin*, if it is induced by a differential operator of geometric origin.

The constructions preserving the geometric origin of an operator have counterparts on the level of Fuchsian systems and monodromy tuples, see Katz (1996) and Dettweiler and Reiter (2007). Furthermore, taking tensor products with rank one systems and middle Hadamard products with rank one systems with exactly two singularities (so-called *Kummer sheaves*) is an invertible operation. Thus a tuple is of geometric origin, if we can produce a tuple of geometric origin out of it, using those invertible operations.

As shown by Katz (1996), a subclass of monodromy tuples of geometric origin are the *linearly rigid* ones with quasi-unipotent monodromy, i.e. those, whose elements are quasi-unipotent, generate an irreducible subgroup in $\mathrm{GL}_n(\mathbb{C})$ and which are, up to simultaneous conjugation, completely determined by the Jordan forms of their elements. In particular, Katz shows that each tuple of this type can be reduced to a geometric tuple of rank one by an iterative sequence of tensor operations with rank one local systems and middle additive convolutions with Kummer sheaves. The most prominent examples of linearly rigid tuples are those induced by hypergeometric differential operators and their generalizations to higher degree and were studied by Riemann (1857), Levelt (1961) and Beukers and Heckman (1989) and many others.

One can extend the notion of rigidity from $\mathrm{GL}_n(\mathbb{C})$ to any reductive complex algebraic group, but then reduction to rank one using Katz methods as in the rigid case fails. Nevertheless, Simpson conjectured that each tuple of this type is of geometric origin, see Simpson (1992).

We know that the elements of the monodromy tuples induced by a fourth order differential Calabi–Yau operator lie in $\mathrm{Sp}_4(\mathbb{C})$. By the discussion above, it seems to be promising to investigate those Calabi–Yau operators inducing an $\mathrm{Sp}_4(\mathbb{C})$-rigid monodromy tuple. A bit surprisingly, the classification of all $\mathrm{Sp}_4(\mathbb{C})$-rigid monodromy tuples reveals the following

**Existence Theorem.** *(Cf. Theorem 3.1.) Each* $\mathrm{Sp}_4(\mathbb{C})$*-rigid tuple consisting of quasi-unipotent elements can be reduced to a tuple of rank one via geometric operations. In particular, it is geometrically constructible using only tuples of rank one and thus of geometric origin.*

Section three of this article is devoted to the proof of the existence theorem via explicit constructions of those tuples using rational pullbacks, tensor and Hadamard products of tuples of rank one. A review of all constructions involved, as well as basic facts concerning rigid monodromy tuples, is given in section two. To construct inducing operators of geometric origin, we translate the constructions to the level of differential operators directly rather than choosing an appropriate cyclic vector of the differential system. This is done in section four. The translation of the construction enables us to compute distinguished solutions of the resulting operators explicitly, which is discussed in section five. Finally, we state an explicit construction of those operators whose induced monodromy tuples have a maximally unipotent element in section six. In the geometric situation, the monodromy tuple lies up to simultaneous conjugation of its elements in $\mathrm{Sp}_4(\mathbb{Z})$. However, we also found potential Calabi–Yau operators, where this is not true for the operator itself, but where the monodromy tuple of its second exterior power lies up to conjugation in $\mathrm{SO}_5(\mathbb{Z})$. In particular, it seems that the second exterior power of a Calabi–Yau operator of order four is a Calabi–Yau operator of order five. We draw the following

**Conjecture.** *An* $\mathrm{Sp}_4(\mathbb{C})$*-rigid tuple consisting of quasi-unipotent elements and having a maximally unipotent element is induced by a differential Calabi–Yau operator if and only if the elements of its second exterior power lie up to simultaneous conjugation in* $\mathrm{SO}_5(\mathbb{Z})$*. Furthermore, the inducing operator is unique.*

The construction of differential operators inducing the remaining monodromy tuples will be done in a subsequent article.

## 2. Rigidity and the middle convolution

### *2.1. Rigidity*

We recall the definition of rigidity in various contexts and state criteria how to read off rigidity via numerical invariants.

**Definition 2.1.**

1. We call $\mathbf{T}$ a *tuple of rank n* if there exist an $r \in \mathbb{N}$ and $T_i \in \mathrm{GL}_n(\mathbb{C})$, $i = 1, \ldots, r+1$ such that $\mathbf{T} = (T_1, \ldots, T_{r+1})$ and $T_1 \cdots T_{r+1} = 1$. Two tuples are equivalent if they are simultaneously conjugate by an element in $\mathrm{GL}_n(\mathbb{C})$.
2. We call a tuple $\mathbf{T}$ *irreducible* of rank $n$ if $\mathbf{T}$ generates an irreducible subgroup $\langle \mathbf{T} \rangle := \langle T_1, \ldots, T_{r+1} \rangle$ of $\mathrm{GL}_n(\mathbb{C})$.
3. We call a tuple $\mathbf{T}$ *quasi-unipotent* if the eigenvalues of all its elements are roots of unity.
4. An irreducible tuple $\mathbf{T}$ is called *symplectic*, resp. *orthogonal*, if $\langle \mathbf{T} \rangle$ respects a skew-symmetric, resp. a symmetric, bilinear form.
5. Let $G \leqslant \mathrm{GL}_n(\mathbb{C})$ be an irreducible reductive algebraic subgroup and $\langle \mathbf{T} \rangle \leqslant G$ be irreducible. We say that $\mathbf{T}$ is $G$*-rigid*, if the following *dimension formula* holds:

$$\sum_{i=1}^{r+1} \operatorname{codim}\bigl(C_G(T_i)\bigr) = 2\bigl(\dim(G) - \dim\bigl(Z(G)\bigr)\bigr),$$

where $C_G(T_i)$ denotes the centralizer of $T_i$ in $G$, the codimension is taken relative to $G$, and $Z(G)$ denotes the center of $G$.

6. We call an irreducible tuple $\mathbf{T}$ of rank $n$ *linearly rigid* if $\mathbf{T}$ is $\mathrm{GL}_n(\mathbb{C})$-rigid and *symplectically rigid* if $\mathbf{T}$ is $\mathrm{Sp}_n(\mathbb{C})$-rigid.

The following lemma stated in Scott (1977) is often helpful to decide whether a tuple $\mathbf{T}$ is reducible.

**Lemma 2.2.** *Let* $\mathbf{T}$ *act on a finite dimensional vector space* $V$. *Then*

$$\sum_{i=1}^{r+1} \mathrm{rk}(T_i - 1) \geqslant \left(\dim(V) - \dim\left(V^{\mathbf{T}}\right)\right) + \left(\dim(V) - \dim\left(V^{\check{\mathbf{T}}}\right)\right),$$

*where* $\check{\mathbf{T}}$ *denotes the tuple corresponding to the dual representation of* $\mathbf{T}$ *and* $V^{\mathbf{T}}$ *the fixed space of* $\mathbf{T}$. *Moreover, if* $\mathbf{T}$ *is irreducible of rank* $n$ *we have*

$$\sum_{i=1}^{r+1} \mathrm{rk}(T_i - 1) \geqslant 2n \quad (\textit{Scott formula}) \quad \textit{and}$$

$$\sum_{i=1}^{r+1} \dim\left(C_{\mathrm{GL}_n(\mathbb{C})}(T_i)\right) \leqslant (r-1)^2 n^2 + 2 \quad (\textit{dimension count}).$$

**Theorem 2.3.**

1. *Let* $\mathbf{T}$ *be irreducible of rank* $n$. *Then* $\mathbf{T}$ *is linearly rigid if and only if* $\mathbf{T}$ *is uniquely determined by the Jordan forms of its elements.*
2. *Let* $\mathbf{T}$ *be an irreducible symplectic tuple of rank* $2m$. *If there exist only finitely many tuples* $(h_1, \ldots, h_{r+1})$ *with* $h_1 \cdots h_{r+1} = 1$ *and such that* $h_i$ *is conjugate in* $\mathrm{Sp}_{2m}(\mathbb{C})$ *to* $T_i$ *then* $\mathbf{T}$ *is* $\mathrm{Sp}_{2m}(\mathbb{C})$*-rigid, i.e., the dimension formula holds.*

**Proof.** The first result goes back to Deligne, Katz and Steenbrink, see e.g. Katz (1996), while the second statement can be found in Strambach and Völklein (1999). □

Alternatively one can consider a tuple as a finite dimensional $\mathbb{C}[F_r]$-module. For this let $F_r$ denote the free group on $r$ generators $f_1, \ldots, f_r$. Setting $f_{r+1} = (f_1 \cdots f_r)^{-1}$ we can view an element in $\mathrm{Mod}(\mathbb{C}[F_r])$ as a pair $(\mathbf{T}, V)$, where $V$ is a finite dimensional vector space over $\mathbb{C}$ and $\mathbf{T} = (T_1, \ldots, T_{r+1})$ is a tuple in $\mathrm{GL}(V)^{r+1}$ such that $f_i$ acts on $V$ via $T_i$ for $i = 1, \ldots, r+1$. We also assign to $\mathbf{T}$ a tuple $\underline{s} = \underline{s}_{\mathbf{T}} = (s_1, \ldots, s_r, s_{r+1} = \infty)$, where $s_1, \ldots, s_r$ are pairwise different elements in $\mathbb{C}$ with an ordering $s_i < s_j$ in $\underline{s}$ if $i < j$.

In a geometric context one can also speak in terms of local systems, as done in the introduction.

### 2.2. Basic properties of the middle convolution

In this section we recall some of the main properties of the middle convolution functor MC. This functor was introduced by Katz (1996) in the category of perverse sheaves. A down to earth version for Fuchsian systems and their monodromy group generators can be found in Dettweiler and Reiter (2007). We recall the main properties of the convolution that are stated in Dettweiler and Reiter (2007, Section 2).

For $(\mathbf{T}, V) \in \mathrm{Mod}(\mathbb{C}[F_r])$, where $\mathbf{T} = (T_1, \ldots, T_{r+1}) \in \mathrm{GL}(V)^{r+1}$, and $\lambda \in \mathbb{C}^\times$ one can construct an element $(C_\lambda(\mathbf{T}), V^r) \in \mathrm{Mod}(\mathbb{C}[F_r])$ as follows. For $k = 1, \ldots, r$, we define $B_k \in \mathrm{GL}(V^r)$ as an element that maps a vector $(v_1, \ldots, v_r)^{\mathrm{tr}} \in V^r$ to

$$\begin{pmatrix} 1 & 0 & & \cdots & & & 0 & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \lambda(T_1-1) & \cdots & \lambda(T_{k-1}-1) & \lambda T_k & (T_{k+1}-1) & \cdots & (T_r-1) & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ 0 & & & \cdots & & & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \\ \vdots \\ \\ \vdots \\ v_r \end{pmatrix}.$$

Further we set $B_{r+1} = (B_1 \cdots B_r)^{-1}$. The subspaces $\mathcal{K} := \bigoplus_{i=1}^r \mathcal{K}_i$, where

$$\mathcal{K}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \ker(T_k - 1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (k\text{-th entry}),\ k = 1, \ldots, r,$$

and

$$L = \bigcap_{k=1}^{r} \ker(B_k - 1) = \ker(B_1 \cdots B_r - 1)$$

of $V^r$ are $\langle B_1, \ldots, B_r \rangle$-invariant. If $\lambda \neq 1$ we have

$$L = \left\langle \begin{pmatrix} T_2 \cdots T_r v \\ T_3 \cdots T_r v \\ \vdots \\ v \end{pmatrix} \;\middle|\; v \in \ker(\lambda \cdot T_1 \cdots T_r - 1) \right\rangle$$

and

$$\mathcal{K} + L = \mathcal{K} \oplus L.$$

**Definition 2.4.** Let $(\mathbf{T}, V) \in \mathrm{Mod}(\mathbb{C}[F_r])$.

1. We call the $\mathbb{C}[F_r]$-module $C_\lambda(V) := (C_\lambda(\mathbf{T}), V^r) := ((B_1, \ldots, B_{r+1}), V)$ the *convolution* of $V$ with $\lambda$, where $\underline{s}_{C_\lambda(\mathbf{T})} := \underline{s}_{\mathbf{T}}$.
2. Let $\mathrm{MC}_\lambda(\mathbf{T}) := (\tilde{B}_1, \ldots, \tilde{B}_{r+1}) \in \mathrm{GL}(V^r/(\mathcal{K} + L))^{r+1}$, where $\tilde{B}_k$ is induced by the action of $B_k$ on $V^r/(\mathcal{K} + L)$. The $K[F_r]$-module $\mathrm{MC}_\lambda(V) := (\mathrm{MC}_\lambda(\mathbf{T}), V^r/(\mathcal{K} + L))$ is called the *middle convolution* of $\mathbf{T}$ with $\lambda$.

**Theorem 2.5.** *Let $(\mathbf{T}, V) \in \mathrm{Mod}(\mathbb{C}[F_r])$ be irreducible. If $\dim(V) = 1$, assume further that at least two of the $T_i$, $i = 1, \ldots, r$, are non-trivial. Let $\lambda \in \mathbb{C}^\times$.*

1. *If $\lambda \neq 1$ then*

$$\dim\bigl(\mathrm{MC}_\lambda(V)\bigr) = \sum_{k=1}^{r} \mathrm{rk}(T_k - 1) - \bigl(\dim(V) - \mathrm{rk}(\lambda \cdot T_1 \cdots T_r - 1)\bigr).$$

2. *If $\lambda_1, \lambda_2 \in \mathbb{C}^\times$ then*

$$\mathrm{MC}_{\lambda_2} \circ \mathrm{MC}_{\lambda_1}(V) \cong \mathrm{MC}_{\lambda_2\lambda_1}(V), \quad \textit{where } \mathrm{MC}_1(V) \cong V.$$

3. $\mathrm{MC}_\lambda(V)$ *is irreducible.*
4. *If* $\mathbf{T}$ *is linearly rigid,* $\mathrm{MC}_\lambda(\mathbf{T})$ *also is.*

Obviously, tensoring a linearly rigid tuple with a rank one tuple preserves linearly rigidity. Nevertheless this operation plays an essential role in the study of linear rigid tuples due to Katz' existence algorithm, see Theorem 2.10.

**Definition 2.6.** Let $(\mathbf{T}_k, V_k) \in \mathrm{Mod}(\mathbb{C}[F_{r_i}])$, $k = 1, 2$, be semisimple and $Set(\underline{s}) = Set(\underline{s}_{\mathbf{T}_1}) \cup Set(\underline{s}_{\mathbf{T}_2})$, $|Set(\underline{s})| = r + 1$, where an ordering on $s_i < s_j$ in $\underline{s}$ is given by the rule: If $s_i, s_j \in Set(\underline{s}_{\mathbf{T}_k})$ then $s_i < s_j$ in $Set(\underline{s}_{\mathbf{T}_k})$ for $k = 1, 2$. Thus we consider $(\mathbf{T}_1, V_1)$ and $(\mathbf{T}_2, V_2)$ as elements in $\mathrm{Mod}(\mathbb{C}[F_r])$, where $T_{k,j} = 1_{V_k}$ if $s_j \notin Set(\underline{s}_{\mathbf{T}_k})$ for $k = 1, 2$. Then we call

$$\mathrm{MT}(V_1, V_2) = V_1 \otimes V_2,$$

$$\mathrm{MT}(\mathbf{T}_1, \mathbf{T}_2) = \mathrm{MT}_{\mathbf{T}_1}(\mathbf{T}_2) = (T_{1,1} \otimes T_{2,1}, \ldots, T_{1,r+1} \otimes T_{2,r+1})$$

the *tensor product* of $(\mathbf{T}_1, V_1)$ and $(\mathbf{T}_2, V_2)$.

**Proposition 2.7.** *Let* $(\mathbf{T}, V) \in \mathrm{Mod}(\mathbb{C}[F_r])$ *be irreducible. If* $\dim(V) = 1$, *assume further that at least two of the* $T_i$, $i = 1, \ldots, r$, *are non-trivial.*

1. *If* $\mathbf{T}$ *is orthogonal, resp. symplectic, then* $\mathrm{MC}_{-1}(\mathbf{T})$ *is symplectic, resp. orthogonal.*
2. *Let* $\mathbf{T}$ *be orthogonal or symplectic and* $\boldsymbol{\Lambda}_1 = (\lambda_1, \lambda_2, (\lambda_1\lambda_2)^{-1})$, $\boldsymbol{\Lambda}_2 = (\lambda_1\lambda_2^{-1}, \lambda_1^{-1}\lambda_2, 1)$ *be rank one tuples such that* $\underline{s}_{\boldsymbol{\Lambda}_1} = \underline{s}_{\boldsymbol{\Lambda}_2} = (s_i, s_j, s_{r+1})$. *Then*
$$\mathrm{MT}_{\boldsymbol{\Lambda}_1^{-1}} \circ \mathrm{MC}_{\lambda_1\lambda_2} \circ \mathrm{MT}_{\boldsymbol{\Lambda}_2} \circ \mathrm{MC}_{(\lambda_1\lambda_2)^{-1}} \circ \mathrm{MT}_{\boldsymbol{\Lambda}_1}(\mathbf{T})$$
*is either orthogonal or symplectic.*

**Proof.** See Dettweiler and Reiter (2000, Corollary 5.10 and Theorem 5.14). □

**Definition 2.8.** Let $\boldsymbol{\Lambda} = (\lambda^{-1}, \lambda)$, $\underline{s}_{\boldsymbol{\Lambda}} = (0, \infty)$, be a rank one tuple. Then we call

$$\mathrm{MH}_\lambda(\mathbf{T}) := \mathrm{MC}_\lambda\big(\mathrm{MT}(\mathbf{T}, \boldsymbol{\Lambda})\big)$$

the *middle Hadamard product* of $\mathbf{T}$ with $\lambda$.

The above definition of the middle Hadamard product is motivated by the fact that the convolution of $f$ with $x^\mu$, $\lambda = \exp(2\pi i\mu)$, can formally be written as a Hadamard product

$$\int f(x)(y-x)^\mu \frac{dx}{y-x} = \int f(x)x^\mu \cdot \left(\frac{y}{x} - 1\right)^{\mu-1} \frac{dx}{x}.$$

Due to the relation between the convolution and the Hadamard product we can switch between this both operations freely.

**Remark 2.9.** Let $\mathbf{T}$ be irreducible and $\lambda \in \mathbb{C}^\times$.
Let $\boldsymbol{\Lambda} = (\lambda, \lambda^{-1})$, $\underline{s}_{\boldsymbol{\Lambda}} = (0, \infty)$, be a rank one tuple. Then

$$\mathrm{MC}_\lambda(\mathbf{T}) = \mathrm{MH}_\lambda\big(\mathrm{MT}(\mathbf{T}, \boldsymbol{\Lambda})\big).$$

The middle convolution yields Katz Existence Theorem, cf. Katz (1996).

**Theorem 2.10.** *Any linearly rigid irreducible tuple* $\mathbf{T}$ *of rank* $n$ *can be reduced to a rank one tuple via a suitable sequence of at most* $n - 1$ *middle convolutions* $\mathrm{MC}_\lambda$ *and tensor products* $\mathrm{MT}_{\boldsymbol{\Lambda}}$ *with rank one tuples* $\boldsymbol{\Lambda}$.

This theorem results in an algorithm to check the existence of a linearly rigid tuple with given Jordan forms. Since MC is multiplicative and $\boldsymbol{\Lambda} \otimes \check{\boldsymbol{\Lambda}}$ is a trivial rank one tuple we can invert each step in the algorithm. The elements of the dual tuple $\check{\boldsymbol{\Lambda}}$ are just the inverse ones of $\Lambda$. Thus we can construct a matrix representation of $\mathbf{T}$.

**Example 2.11.** The tuple

$$\mathbf{T} = (T_0, T_1, T_\infty) = \mathrm{MH}_\beta \circ \mathrm{MH}_{\beta^{-1}} \circ \mathrm{MH}_\alpha\bigl(1, \alpha, \alpha^{-1}\bigr), \quad \alpha, \beta \in \mathbb{C}^* \setminus \{1\}$$

is a symplectic tuple of rank four. Using the methods described in this section we can compute $\mathbf{T}$ explicitly. Setting $A = \alpha + \alpha^{-1} - 2$, $B = \beta + \beta^{-1} - 2$ we get

$$T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ AB & AB & A+B & 1 \end{pmatrix}.$$

This is a special case of a monodromy tuple of a generalized hypergeometric differential equation. Those monodromy tuples were first described by Levelt (1961). For a detailed study of the monodromy we refer to the paper Beukers and Heckman (1989).

## 2.3. The numerology of the middle convolution

We recall the effect of the middle convolution on the Jordan forms of the local monodromy, given by Katz (1996, Chapter 6):

For $i = 1, \ldots, r+1$, we write $\mathbf{J}(T_i) = \bigoplus_{\rho \in \mathbb{C}} \bigoplus_j \rho \mathbf{J}(j)^{v(i,\rho,j)}$, $v(i,\rho,j) \in \mathbb{N}_0$, as a direct sum of Jordan blocks $\rho\mathbf{J}(j)$ of size $j$ with respect to the eigenvalue $\rho$ with multiplicity $v(i,\rho,j)$. We also write $T_0$, resp. $T_\infty$, for the monodromy at 0, resp. $\infty$.

**Proposition 2.12.** *Let $\mathbf{T}$ be irreducible of rank $n$ and $\lambda \neq 1$. The transformation of the Jordan forms of its elements under the middle convolution is given by*

$$\begin{aligned} \mathbf{J}\bigl(\mathrm{MC}_\lambda(T_i)\bigr) = {} & \bigoplus_{\rho \in \mathbb{C}\setminus\{1,\lambda^{-1}\}} \bigoplus_j \lambda\rho\mathbf{J}(j)^{v(i,\rho,j)} \bigoplus_{j \geqslant 2} \lambda\mathbf{J}(j-1)^{v(i,1,j)} \\ & \bigoplus_j \mathbf{J}(j+1)^{v(i,\lambda^{-1},j)} \bigoplus \mathbf{J}(1)^{k_i} \quad (i = 1, \ldots, r), \\ \mathbf{J}\bigl(\mathrm{MC}_\lambda(T_{r+1})\bigr) = {} & \bigoplus_{\rho \in \mathbb{C}\setminus\{1,\lambda\}} \bigoplus_j \lambda^{-1}\rho\mathbf{J}(j)^{v(r+1,\rho,j)} \bigoplus_j \mathbf{J}(j-1)^{v(r+1,\lambda,j)} \\ & \bigoplus_j \lambda^{-1}\mathbf{J}(j+1)^{v(r+1,1,j)} \bigoplus \lambda^{-1}\mathbf{J}(1)^{k_{r+1}}, \end{aligned}$$

*where $k_j$ is determined by*

$$\mathrm{rk}\bigl(\mathrm{MC}_\lambda(\mathbf{T})\bigr) = \sum_{i=1}^{r} \mathrm{rk}(T_i - 1) + \mathrm{rk}\bigl(\lambda^{-1}T_\infty - 1\bigr) - n.$$

This also shows that the middle convolution $\mathrm{MC}_\lambda$ preserves linear rigidity, cf. Theorem 2.5.

From the definition of the middle Hadamard product and the above proposition we can derive the Jordan forms of $\mathrm{MH}_\lambda(\mathbf{T})$:

**Proposition 2.13.** *Let $\mathbf{T}$ be irreducible of rank $n$ and $\lambda \neq 1$. The transformation of the Jordan forms of its elements under the middle Hadamard product is given by*

$$\begin{aligned}
\mathbf{J}\big(\mathrm{MH}_\lambda(T_i)\big) = {} & \bigoplus_{\rho\in\mathbb{C}\setminus\{1,\lambda^{-1}\}} \bigoplus_j \lambda\rho\mathbf{J}(j)^{\nu(i,\rho,j)} \bigoplus_j \mathbf{J}(j+1)^{\nu(i,\lambda^{-1},j)} \\
& \bigoplus_{j\geqslant 2} \lambda\mathbf{J}(j-1)^{\nu(i,1,j)} \bigoplus \mathbf{J}(1)^{k_i} \quad (i\neq 0, r+1), \\
\mathbf{J}\big(\mathrm{MH}_\lambda(T_0)\big) = {} & \bigoplus_{\rho\in\mathbb{C}\setminus\{1,\lambda\}} \bigoplus_j \rho\mathbf{J}(j)^{\nu(0,\rho,j)} \bigoplus_j \mathbf{J}(j+1)^{\nu(0,1,j)} \\
& \bigoplus_{j\geqslant 2} \lambda\mathbf{J}(j-1)^{\nu(0,\lambda,j)} \bigoplus \mathbf{J}(1)^{k_0}, \\
\mathbf{J}\big(\mathrm{MH}_\lambda(T_{r+1})\big) = {} & \bigoplus_{\rho\in\mathbb{C}\setminus\{1,\lambda^{-1}\}} \bigoplus_j \rho\mathbf{J}(j)^{\nu(r+1,\rho,j)} \quad \bigoplus_{j\geqslant 2} \mathbf{J}(j-1)^{\nu(r+1,1,j)} \\
& \bigoplus_j \lambda^{-1}\mathbf{J}(j+1)^{\nu(r+1,\lambda^{-1},j)} \bigoplus \lambda^{-1}\mathbf{J}(1)^{k_{r+1}}
\end{aligned}$$

*where $k_j$ is determined by*

$$\mathrm{rk}\big(\mathrm{MH}_\lambda(\mathbf{T})\big) = \sum_{T_i\neq T_0} \mathrm{rk}(T_i - 1) + \mathrm{rk}\big(\lambda^{-1}T_0 - 1\big) - n.$$

## 3. Classification of symplectically rigid tuples of rank four

This section is devoted to the classification of symplectically rigid tuples of rank four. In particular we show.

**Theorem 3.1.** *Let* $\mathbf{T}$ *be a symplectically rigid tuple of rank four consisting of quasi-unipotent elements. Then* $\mathbf{T}$ *is coming from geometry, i.e.* $\mathbf{T}$ *is a monodromy tuple of a factor of a Picard–Fuchs equation. Moreover it can be constructed by a sequence of geometric operations starting with a rank one tuple. These geometric operations include tensor products, rational pullbacks and the middle convolution.*

Roughly speaking the proof of Theorem 3.1 is based on the following steps:

*Step one*: Using Theorem 2.3 we classify in Table 2 all possible symplectically rigid irreducible tuples $\mathbf{T}$ of rank four via the tuples

$$P_i := \big(\dim C_{\mathrm{Sp}_4(\mathbb{C})}(T_1), \dots, \dim C_{\mathrm{Sp}_4(\mathbb{C})}(T_{r+1})\big)$$

of the centralizer dimensions of their elements. We list these centralizer dimensions in Table 1. Via Möbius transformations, which act sharply 3-transitive, and more generally the action of the Artin braid group $\mathcal{B}_r$ on $\mathbf{T}$ that permutes the local monodromies, we can order the entries according to increasing dimensions. Thus we get the finite list $P_1, \dots, P_5$ in Table 2. Further, we refine these cases by the subcases

$$P_i\big(\dim C_{\mathrm{GL}_4(\mathbb{C})}(T_1), \dots, \dim C_{\mathrm{GL}_4(\mathbb{C})}(T_{r+1})\big).$$

E.g., the $P_3(4,8,10,10)$ case denotes irreducible quadruples $\mathbf{T}$ with

$$\big(\dim C_{\mathrm{Sp}_4(\mathbb{C})}(T_1), \dots, \dim C_{\mathrm{Sp}_4(\mathbb{C})}(T_4)\big) = (2,6,6,6)$$

and

$$\big(\dim C_{\mathrm{GL}_4(\mathbb{C})}(T_1), \dots, \dim C_{\mathrm{GL}_4(\mathbb{C})}(T_4)\big) = (4,8,10,10).$$

Moreover Table 1 shows that

$$\mathbf{J}(T_1) \in \big\{\pm\mathbf{J}(4), \big(-\mathbf{J}(2), \mathbf{J}(2)\big), \big(x\mathbf{J}(2), x^{-1}\mathbf{J}(2)\big), \big(x, y, y^{-1}, x^{-1}\big)\big\},$$

$\mathbf{J}(T_2) = (-1,-1,1,1)$ and $\mathbf{J}(T_3) = \mathbf{J}(T_4) = (\mathbf{J}(2), 1, 1)$.

**Table 1**
The Jordan forms of elements in $\mathrm{Sp}_4(\mathbb{C})$ and $\mathrm{SO}_5(\mathbb{C}) = \Lambda^2\,\mathrm{Sp}_4(\mathbb{C})$.

| Jordan form in $\mathrm{Sp}_4(\mathbb{C})$ | Jordan form in $\mathrm{SO}_5(\mathbb{C})$ | Centr. dimension in $\mathrm{Sp}_4(\mathbb{C})$ | Centr. dimension in $\mathrm{GL}_4(\mathbb{C})$ | Conditions |
|---|---|---|---|---|
| $\pm(1,1,1,1)$ | $(1,1,1,1,1)$ | 10 | 16 | |
| $\pm(\mathbf{J}(2),1,1)$ | $(\mathbf{J}(2),\mathbf{J}(2),1)$ | 6 | 10 | |
| $\pm(\mathbf{J}(2),\mathbf{J}(2))$ | $(\mathbf{J}(3),1,1)$ | 4 | 8 | |
| $\pm\mathbf{J}(4)$ | $\mathbf{J}(5)$ | 2 | 4 | |
| $(-1,-1,1,1)$ | $(-1,-1,-1,-1,1)$ | 6 | 8 | |
| $\pm(-\mathbf{J}(2),1,1)$ | $(-\mathbf{J}(2),-\mathbf{J}(2),1)$ | 4 | 6 | |
| $(-\mathbf{J}(2),\mathbf{J}(2))$ | $(-\mathbf{J}(3),-1,1)$ | 2 | 4 | |
| $(x,x,x^{-1},x^{-1})$ | $(x^2,1,1,1,x^{-2})$ | 4 | 8 | $x^2\neq 1$ |
| $(x,1,1,x^{-1})$ | $(x,x,1,x^{-1},x^{-1})$ | 4 | 6 | $x^2\neq 1$ |
| $(x\mathbf{J}(2),x^{-1}\mathbf{J}(2))$ | $(\mathbf{J}(3),x^2,x^{-2})$ | 2 | 4 | $x^2\neq 1$ |
| $(x,x^{-1},\mathbf{J}(2))$ | $(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),1)$ | 2 | 4 | $x^2\neq 1$ |
| $(x,y,y^{-1},x^{-1})$ | $(xy,xy^{-1},1,x^{-1}y,x^{-1}y^{-1})$ | 2 | 4 | $x^2,y^2\neq 1$<br>$x\neq y^{\pm 1}$ |

**Table 2**
The centralizer conditions for symplectically rigid tuples.

| Case | Subcases | Remarks |
|---|---|---|
| $P_1$ | $(4,4,10)$ | lin. rigid |
| $(2,2,6)$ | $(4,4,8)$ | $\Lambda^2$ lin. rigid |
| $P_2$ | $(4,6,6)$ | |
| $(2,4,4)$ | $(4,6,8)$ | lin. rigid |
| | $(4,8,8)$ | red. (dimension count) |
| $P_3$ | $(4,10,10,10)$ | red. (Scott) |
| $(2,6,6,6)$ | $(4,8,10,10)$ | |
| | $(4,8,8,10)$ | $\Lambda^2$ red. |
| | $(4,8,8,8)$ | $\Lambda^2$ red. |
| $P_4$ | $(8,8,10,10)$ | red. (dimension count) |
| $(4,4,6,6)$ | $(6,8,10,10)$ | lin. rigid |
| | $(6,6,10,10)$ | |
| | $(8,8,8,10)$ | lin. rigid |
| | $(6,8,8,10)$ | |
| | $(6,6,8,10)$ | $\Lambda^2$ red. |
| | $(8,8,8,8)$ | $\Lambda^2$ red. |
| | $(6,8,8,8)$ | $\Lambda^2$ red. |
| | $(6,6,8,8)$ | $\Lambda^2$ lin. rigid |
| $P_5$ | $(10,10,10,10,10)$ | lin. rigid |
| $(6,6,6,6,6)$ | $(8,10,10,10,10)$ | red. (Scott) |
| | $(8,8,10,10,10)$ | red. (Scott) |
| | $(8,8,8,10,10)$ | $\Lambda^2$ lin. rigid |
| | $(8,8,8,8,10)$ | $\Lambda^2$ red. |
| | $(8,8,8,8,8)$ | $\Lambda^2$ red. |

*Step two*: The irreducibility condition restricts the possible tuples of Jordan forms via the Scott formula or the dimension count in Lemma 2.2. E.g. there is no rigid tuple of rank four with Jordan forms

$$\big(\mathbf{J}(4),\mathbf{J}(4),\big(\mathbf{J}(2),1,1\big)\big)$$

in the $P_1(4,4,10)$ case, as $7=\sum_i \mathrm{rk}(T_i-1)<2\cdot 4$.

*Step three*: We check whether $\mathbf{T}$ is linearly rigid using the dimension count in Theorem 2.3. In the positive case the claim follows from Katz' algorithm, see Theorem 2.10. Moreover the algorithm imposes the conditions for the existence of such a $\mathbf{T}$ depending on the eigenvalues of the $T_i$.

*Step four*: Using the operations in Proposition 2.7 we try to construct a tuple $\tilde{\mathbf{T}}$ in an orthogonal group of dimension 3, 4, 5 or 6. Due to the exceptional isomorphisms we have

$$\mathrm{Sym}^2 \mathrm{Sp}_2(\mathbb{C}) = \mathrm{SO}_3(\mathbb{C}), \qquad \mathrm{Sp}_2(\mathbb{C}) \otimes \mathrm{Sp}_2(\mathbb{C}) = \mathrm{SO}_4(\mathbb{C}),$$
$$\Lambda^2 \mathrm{Sp}_4(\mathbb{C}) = \mathrm{SO}_5(\mathbb{C}), \qquad \Lambda^2 \mathrm{SL}_4(\mathbb{C}) = \mathrm{SO}_6(\mathbb{C}),$$

which can again result in linearly rigid tuples. E.g. an irreducible orthogonal triple $\mathbf{T}$ of rank three with $\mathbf{J}(\mathbf{T}) = (\mathbf{J}(3), \mathbf{J}(3), \mathbf{J}(3))$ yields a linearly rigid triple $\tilde{\mathbf{T}}$ of rank two with $\mathbf{J}(\tilde{\mathbf{T}}) = (\mathbf{J}(2), \mathbf{J}(2), -\mathbf{J}(2))$.

It turns out that in all $P_i$ cases we either get contradictions to the irreducibility or we end up with a rank one tuple. In the latter case we obtain a suitable sequence of operations that allows us to construct this symplectically rigid tuple $\mathbf{T}$ of rank four, since each operation is invertible.

Moreover if the symplectically rigid tuple of rank four is quasi-unipotent it turns out that it can be constructed using only geometric operations, like Hadamard products, rational pullbacks and tensor products, cf. André (1989, Chapter II).

We begin with *Step one* and classify the Jordan forms in $\mathrm{Sp}_4(\mathbb{C})$ and their centralizer dimensions. Since $\Lambda^2 \mathrm{Sp}_4(\mathbb{C}) = \mathrm{SO}_5(\mathbb{C})$ we also determine the Jordan forms in $\mathrm{SO}_5(\mathbb{C})$.

In the following sections we rearrange the order of the centralizer dimensions in Table 2 via Möbius transformations to simplify the proofs. If $\mathbf{T}$ is a triple we can assume that $\underline{s}_{\mathbf{T}} = \{0, 1, \infty\}$. Thus we also index $\mathbf{T} = (T_0, T_1, T_\infty)$. E.g., a linearly rigid tuple in the $P_1(4, 10, 4)$ case such that $T_0$ is unipotent can be written as a sequence of three Hadamard products starting from a rank one tuple, see Example 2.11. However, in the $P_1(4, 4, 10)$ case the Katz algorithm requires additional tensor products with rank one tuples.

To abbreviate the notations we denote by $\mathbf{J}(\mathbf{T})$ the tuple of Jordan forms. Further we write $\mathbf{J}_s(\mathbf{T})$ for $(\mathbf{J}_s(T_1), \ldots, \mathbf{J}_s(T_{r+1}))$, where $\mathbf{J}_s(T_i)$ denotes the semisimple part of $\mathbf{J}(T_i)$.

### 3.1. *The $P_1$ case*

#### 3.1.1. *The $P_1(4, 10, 4)$ case*

**Remark 3.2.** We omit the linearly rigid $P_1(4, 10, 4)$ case. This well-studied case corresponds to monodromy tuples of generalized hypergeometric differential equations of order four and is settled by Katz' algorithm. For an example where $T_0$ is maximally unipotent, see Example 2.11.

#### 3.1.2. *The $P_1(4, 8, 4)$ case*

**Theorem 3.3.** *A symplectically rigid tuple $\mathbf{T}$ in the case $P_1(4, 8, 4)$ can be obtained from a rank one tuple using the middle Hadamard product and tensor products. Moreover, the tuple $\mathbf{T}$ can be written as*

$$\mathbf{T} = \mathrm{MH}_{-1}\big(\Lambda^2(\mathbf{S})\big),$$

*where $\mathbf{S}$ is a linearly rigid rank four triple containing a transvection.*

**Proof.** By Theorem 2.5 and Proposition 2.7 the Hadamard product $\mathrm{MH}_{-1}(\mathbf{T})$ yields an irreducible orthogonal triple of rank $m$, where

$$m = \mathrm{rk}(-T_0 - 1) + \mathrm{rk}(T_1 - 1) + \mathrm{rk}(T_\infty - 1) - 4 \in \{4, 5, 6\}.$$

Hence we can apply one of the identities

$$\Lambda^2 \mathrm{Sp}_4(\mathbb{C}) = \mathrm{SO}_5(\mathbb{C}), \qquad \Lambda^2 \mathrm{SL}_4(\mathbb{C}) = \mathrm{SO}_6(\mathbb{C})$$

to obtain a triple of rank four containing a transvection, since by Proposition 2.13

$$\mathbf{J}\big(\mathrm{MH}_{-1}(T_1)\big) = \big(\mathbf{J}(2)^2, \mathbf{J}(1)^{m-4}\big).$$

For $m = 4$ we use the natural embedding of $\mathrm{GO}_4(\mathbb{C})$ in $\mathrm{SO}_5(\mathbb{C})$. Thus the triple is linearly rigid and the claim follows from Katz' algorithm. □

**Remark 3.4.** The construction of $\mathbf{T}$ is in general not unique. In the above case one could also get $\mathbf{T}$ by using that $\Lambda^2(\mathbf{T})$ yields a linearly rigid tuple and then apply Katz' algorithm. However in this construction the computation of the matrix representation of $\mathbf{T}$ is more complicated.

**Corollary 3.5.** *Let $\mathbf{T}$ be as in Theorem 3.3 such that $T_0$ is maximally unipotent and $\mathbf{J}_s(T_\infty) = (xy, xy^{-1}, x^{-1}y, (xy)^{-1})$. Then*

$$T_0 = \begin{pmatrix} 1 & ab & 0 & (a+b)^2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -ab \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} -1 & -2ab & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & ab & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix},$$

*where $a = x + \frac{1}{x}$, $b = y + \frac{1}{y}$, $x, y \in \mathbb{C}^*$ and $ab \neq 0$. The tuple $\mathbf{T}$ can be obtained as follows:*

$$\mathbf{T} = \mathrm{MH}_{-1} \circ \mathrm{MT}_{\boldsymbol{\Lambda}_1}\left(\Lambda^2\mathbf{S}\right), \quad \textit{where}$$
$$\mathbf{S} = \mathrm{MH}_{(ix)} \circ \mathrm{MH}_{-(ix)^{-1}} \circ \mathrm{MT}_{\boldsymbol{\Lambda}_1} \circ \mathrm{MH}_{-iy}(\boldsymbol{\Lambda}_0)$$

*with $\boldsymbol{\Lambda}_0 = (1, (iy)^{-1}, iy)$ and $\boldsymbol{\Lambda}_1 = (-1, 1, -1)$ are a rank one triples. Further, $\mathrm{MT}_{(i,1,i^{-1})}\mathbf{S}$ is symplectic and linearly rigid of rank four with*

$$\left(i\mathbf{J}(S_0), \mathbf{J}(S_1), -i\mathbf{J}_s(S_\infty)\right) = \left(\left(i\mathbf{J}(2), -i\mathbf{J}(2)\right), \left(\mathbf{J}(2), 1, 1\right), \left(x, y, y^{-1}, x^{-1}\right)\right).$$

**Proof.** The tuple $\mathbf{T}$ can be constructed using the matrices in Section 2.2 according to the given sequence of Hadamard products and tensor products. Proposition 2.13 allows to keep track of the change of the Jordan forms under the Hadamard product. We demonstrate this for the case, where $x, x^{-1}, y, y^{-1}$ are pairwise different: We start with a rank one triple $\boldsymbol{\Lambda}_0 = (1, (iy)^{-1}, iy)$ and apply $\mathrm{MH}_{-iy}$. This yields a rank two triple with Jordan forms $(\mathbf{J}(2), (-1, 1), (-iy^{-1}, -iy))$. Then we proceed with the tensor product $\mathrm{MT}_{\boldsymbol{\Lambda}_1}$ and so on. Tabulating the operations and the change of the Jordan forms we get

| rk | Operation | Jordan | Forms | |
|---|---|---|---|---|
| 1 | | $(1)$ | $(iy)$ | $((iy)^{-1})$ |
| 2 | $\mathrm{MH}_{-iy}$ | $\mathbf{J}(2)$ | $(-1, 1)$ | $(-iy^{-1}, -iy)$ |
| 2 | $\mathrm{MT}_{\boldsymbol{\Lambda}_1}$ | $-\mathbf{J}(2)$ | $(-1, 1)$ | $(iy^{-1}, iy)$ |
| 3 | $\mathrm{MH}_{-(ix)^{-1}}$ | $(-\mathbf{J}(2), 1)$ | $((ix)^{-1}, 1, 1)$ | $(iy^{-1}, iy, ix^{-1})$ |
| 4 | $\mathrm{MH}_{(ix)}$ | $(-\mathbf{J}(2), \mathbf{J}(2))$ | $(\mathbf{J}(2), 1, 1)$ | $(iy^{-1}, iy, ix^{-1}, ix)$ |
| 5 | $\Lambda^2$ | $(-\mathbf{J}(3), 1, 1)$ | $(\mathbf{J}(2), \mathbf{J}(2), 1)$ | $-(xy^{-1}, xy, 1, (xy)^{-1}, x^{-1}y)$ |
| 5 | $\mathrm{MT}_{\boldsymbol{\Lambda}_1}$ | $(\mathbf{J}(3), -1, -1)$ | $(\mathbf{J}(2), \mathbf{J}(2), 1)$ | $(xy^{-1}, xy, 1, x^{-1}y^{-1}, x^{-1}y)$ |
| 4 | $\mathrm{MH}_{-1}$ | $\mathbf{J}(4)$ | $(-1, -1, 1, 1)$ | $(xy^{-1}, xy, x^{-1}y^{-1}, x^{-1}y)$ |

By Proposition 2.7 we know that $\mathrm{MT}_{(i,1,i^{-1})}(\mathbf{S})$ is symplectic and we use that $\Lambda^2\,\mathrm{Sp}_4(\mathbb{C}) = \mathrm{SO}_5(\mathbb{C})$. In the general case the Jordan form of the third element in each step is obtained by replacing $k$ equal eigenvalues $z$ by $z\mathbf{J}(k)$.

The conditions for the irreducibility follow from the fact that the middle Hadamard product has to be non-trivial in each step, i.e. $i \neq \pm x, \pm y$ by Theorem 2.5. Thus $ab \neq 0$. □

**Corollary 3.6.** *Let $\mathbf{T}$ be as in Corollary 3.5. Then the Zariski closure of $\langle\mathbf{T}\rangle$ is $\mathrm{Sp}_4(\mathbb{C})$. Moreover if $ab, a^2 + b^2 \in \mathbb{Z}$ then $\langle\mathbf{T}\rangle$ is contained up to conjugation in $\mathrm{Sp}_4(\mathbb{Z})$. Further, if $\mathbf{T}$ is quasi-unipotent then the conditions are also necessary.*

**Proof.** Since $\mathbf{J}(T_1) = (-1, -1, 1, 1)$ the Zariski closure of $\langle\mathbf{T}\rangle$ is not $\mathrm{Sym}^3(\mathrm{SL}_2(\mathbb{C}))$ and the first statement follows from Corollary A.3. The matrix representation shows that the conditions are sufficient.

The necessary condition for the group $\langle \mathbf{T} \rangle$ to be contained in $\mathrm{Sp}_4(\mathbb{Z})$ is that all traces of all elements are integers. Hence

$$\mathrm{tr}(T_\infty) = ab, \qquad \mathrm{tr}\big(T_\infty^2\big) = \big(a^2-2\big)\big(b^2-2\big) = (ab)^2 + 4 - 2\big(a^2+b^2\big) \in \mathbb{Z}.$$

Hence $ab, 2(a^2+b^2) \in \mathbb{Z}$. But if $a$, $b$ are sums of roots of unity then $2(a^2+b^2) \in \mathbb{Z}$ implies $(a^2 + b^2) \in \mathbb{Z}$. $\square$

### 3.2. The $P_2$ case

#### 3.2.1. The $P_2(4,6,6)$ case

**Theorem 3.7.** *Let* $\mathbf{T}$ *be a symplectically rigid tuple in the case* $P_2(4,6,6)$*, where*

$$\mathbf{J}_s(\mathbf{T}) = \big(\big(z_1z_2, z_1z_2^{-1}, z_1^{-1}z_2, (z_1z_2)^{-1}\big), \big(1, -x^2, -x^{-2}, 1\big), \big(y^2, -1, -1, y^{-2}\big)\big),$$

*with* $x, y, z_1, z_2 \in \mathbb{C}^*$*. Then* $\mathbf{T}$ *can be written*

$$\mathbf{T} = \mathrm{MH}_{-1}\big(\mathrm{MT}(\mathbf{S}_1, \mathbf{S}_2)\big), \quad \textit{where } \mathbf{S}_i = \mathrm{MT}_{\boldsymbol{\Lambda}_{2i}}(\mathrm{MH}_{z_i x y^{-1}} \boldsymbol{\Lambda}_{1i})$$

*with* $\boldsymbol{\Lambda}_{2i} = (z_i^{-1}, x^{-1}, z_i x)$*,* $\boldsymbol{\Lambda}_{1i} = (z_i^2, z_i^{-1}xy, (z_i xy)^{-1})$*,* $i = 1, 2$*.*

**Proof.** The tuple

$$\mathbf{S} = \mathrm{MT}_{\boldsymbol{\Lambda}} \circ \mathrm{MC}_{-1}(\mathbf{T}), \quad \boldsymbol{\Lambda} = (-1, 1, -1),$$

is an orthogonal triple of rank

$$m = \mathrm{rk}(T_0 - 1) + \mathrm{rk}(T_1 - 1) + \mathrm{rk}(-T_\infty - 1) - 4 \in \{3, 4\}$$

by Theorem 2.5 and Proposition 2.7. Using that

$$\mathrm{SO}_4(\mathbb{C}) = \mathrm{Sp}_2(\mathbb{C}) \otimes \mathrm{Sp}_2(\mathbb{C}), \qquad \mathrm{SO}_3(\mathbb{C}) = \mathrm{Sym}^2\,\mathrm{Sp}_2(\mathbb{C})$$

we can write $\mathbf{S}$ as $\mathbf{S} = \mathbf{S}_1 \otimes \mathbf{S}_2$ with

$$\big(\mathbf{J}(S_{i0}), \mathbf{J}(S_{i1}), \mathbf{J}(S_{i\infty})\big) = \big(\big(z_i, z_i^{-1}\big), \big(x, x^{-1}\big), \pm\big(y, y^{-1}\big)\big), \quad i = 1, 2.$$

Since $\mathbf{S}_1$ and $\mathbf{S}_2$ are linearly rigid the claim follows from Katz' algorithm. $\square$

**Corollary 3.8.** *Let* $\mathbf{T}$ *be as in Theorem 3.7, such that* $T_0$ *is maximally unipotent. Then*

$$T_0 = \begin{pmatrix} 1 & -a+b & a & -2 \\ 0 & 1 & -2 & b \\ 0 & 0 & 1 & a-b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -a \\ 2 & a+b & a & -a^2+1 \end{pmatrix},$$

*where* $a = x + \frac{1}{x}$*,* $b = y + \frac{1}{y}$ *and* $a \neq b$*. The tuple* $\mathbf{T}$ *can be written as*

$$\mathbf{T} = \mathrm{MH}_{-1}\big(\mathrm{Sym}^2\,\mathbf{S}\big), \quad \mathbf{S} = \mathrm{MT}_{\boldsymbol{\Lambda}} \circ \mathrm{MH}_{xy^{-1}}(\boldsymbol{\Lambda}_0),$$

*where* $\boldsymbol{\Lambda} = (1, x^{-1}, x)$ *and* $\boldsymbol{\Lambda}_0 = (1, xy, (xy)^{-1})$ *are rank one triples with* $\underline{s}_{\boldsymbol{\Lambda}} = \underline{s}_{\boldsymbol{\Lambda}_0} = (0, 1, \infty)$*.*

**Proof.** The proof is analogous to the proof of Corollary 3.5. $\square$

**Corollary 3.9.** *Let* $\mathbf{T}$ *be as in Corollary 3.8. Then the Zariski closure of* $\langle \mathbf{T} \rangle$ *is* $\mathrm{Sp}_4(\mathbb{C})$ *if and only if* $a^2 \neq 1$ *and* $b^2 \neq 1$*. The generated group is up to conjugation contained in* $\mathrm{Sp}_4(\mathbb{Z})$ *if and only if* $a^2, b^2, ab \in \mathbb{Z}$*.*

**Proof.** By construction there are at most two symplectically rigid tuples with given Jordan forms since $\mathrm{Sym}^2$ does not act bijectively on the Jordan forms. However if $a = -b$ then the Jordan forms determine the tuple $\mathbf{T}$ uniquely since a rank two triple with Jordan forms $(\mathbf{J}(2), (x, x^{-1}), (x, x^{-1}))$ is reducible.

Further if $a^2 = b^2 = 1$ then $x$, $y$ are sixth roots of unity and $\mathbf{T}$ can be also written as $\mathrm{Sym}^3$ of a rank two tuple. By uniqueness and Corollary A.3 the first claim follows.

If the generated group is up to conjugation contained in $\mathrm{Sp}_4(\mathbb{Z})$ then the trace condition implies $a^2, b^2 \in \mathbb{Z}$. By construction the middle convolution $\mathrm{MC}_{-1}$ and taking $\mathrm{Sym}^2$ are compatible with the action of a field automorphism. Thus if $ab \notin \mathbb{Z}$ then there exists a $\sigma \in \mathrm{Gal}(\mathbb{Q}(a,b)/\mathbb{Q})$ such that $\sigma(a) = a$ and $\sigma(b) = -b$. But then we get $\mathbf{T}^\sigma = \mathbf{T}$ and $\mathbf{S}^\sigma \neq \mathbf{S}$, a contradiction. The matrix representation shows that these conditions are also sufficient. Namely, if $a, b \notin \mathbb{Z}$, but $ab \in \mathbb{Z}$ then $a = n_1\sqrt{d}$ and $b = n_2\sqrt{d}$. Thus if we conjugate the matrices in Corollary 3.8 by $\mathrm{diag}(\sqrt{d}, 1, 1, \sqrt{d})$ we get a representation in $\mathrm{Sp}_4(\mathbb{Z})$. □

### 3.2.2. The $P_2(4,6,8)$ case

Since the proofs of the statements in the linearly rigid $P_2(4,6,8)$ case are analogous to the proofs before we omit them.

**Theorem 3.10.** *A linearly rigid tuple $\mathbf{T}$ in the case $P_2(4,6,8)$, where*

$$\mathbf{J}_s(\mathbf{T}) = \left(\left(z_1, z_2, z_2^{-1}, z_1^{-1}\right), \left(1, 1, y, y^{-1}\right), \left(x, x, x^{-1}, x^{-1}\right)\right),$$

*can be obtained as*

$$\mathbf{T} = \mathrm{MT}_{\boldsymbol{\Lambda}_3} \circ \mathrm{MH}_{xz_1} \circ \mathrm{MT}_{\boldsymbol{\Lambda}_2} \circ \mathrm{MH}_{(xz_1)^{-1}} \circ \mathrm{MT}_{\boldsymbol{\Lambda}_1} \circ \mathrm{MH}_{yz_1z_2}(\boldsymbol{\Lambda}_0),$$

*where $\boldsymbol{\Lambda}_3 = (z_1^{-1}, 1, z_1)$, $\boldsymbol{\Lambda}_2 = (z_1^2, 1, z_1^{-2})$, $\boldsymbol{\Lambda}_1 = ((z_1z_2)^{-1}, y^{-1}, yz_1z_2)$ and $\boldsymbol{\Lambda}_0 = (z_2^2, y(z_1z_2)^{-1}, z_1z_2^{-1}y^{-1})$.*

**Corollary 3.11.** *Let $\mathbf{T}$ be as in Theorem 3.10 such that $T_0$ is maximally unipotent. Then*

$$T_0 = \begin{pmatrix} 1 & -1 & 0 & a-2 \\ 0 & 1 & a-2 & 0 \\ 0 & 0 & 1 & -b+2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & b-2 \\ 1 & 0 & 1 & b-1 \end{pmatrix},$$

*where $a = x + \frac{1}{x}$, $b = y + \frac{1}{y}$, $x, y \in \mathbb{C}^* \setminus \{1\}$. The tuple $\mathbf{T}$ can be obtained via*

$$\mathbf{T} = \mathrm{MH}_x \circ \mathrm{MH}_{x^{-1}} \circ \mathrm{MT}_{\boldsymbol{\Lambda}_1} \circ \mathrm{MH}_y(\boldsymbol{\Lambda}_0),$$

*where $\boldsymbol{\Lambda}_1 = (1, y^{-1}, y)$ and $\boldsymbol{\Lambda}_0 = (1, y, y^{-1})$ are rank one triples.*

**Corollary 3.12.** *Let $\mathbf{T}$ be as in Corollary 3.11. Then $\langle \mathbf{T} \rangle$ is contained up to conjugation in $\mathrm{Sp}_4(\mathbb{Z})$ if and only if $a, b \in \mathbb{Z}$. The Zariski closure of $\langle \mathbf{T} \rangle$ is $\mathrm{Sp}_4(\mathbb{C})$ if and only if $a \neq 0$ and $b \neq -1$.*

## 3.3. The $P_3$, $P_4$ and $P_5$ cases

In this section we show that in the cases $P_3$, $P_4$ and $P_5$ all symplectically rigid tuples $\mathbf{T}$ can be reduced via geometric operations to rank one tuples. Since we prefer to work with the convolution we index $\mathbf{T} = (T_1, \ldots, T_r, T_{r+1} = T_\infty)$. In order to shortcut the following proofs we use without citing that the application of $\mathrm{MC}_{-1}$ changes a symplectical tuple into an orthogonal one by Proposition 2.7 whose rank is given by Theorem 2.5. Moreover, due to Katz' algorithm it suffices to relate $\mathbf{T}$ to a linearly rigid tuple.

### 3.3.1. The $P_3$ case

**Theorem 3.13.** *In all the $P_3$ cases a symplectically rigid tuple $\mathbf{T}$ can be reduced via middle convolution operations, taking tensor products and rational pullbacks to a rank one tuple. Further there exists no $\mathbf{T}$ with a maximally unipotent element.*

**Proof.**

1. The case $P_3(4,10,10,10)$ is ruled out by the Scott formula.
2. In the case $P_3(4,8,10,10)$ the Scott formula implies that $\mathrm{rk}(T_1-1)=\mathrm{rk}(T_1+1)=4$. Let $\boldsymbol{\Lambda}_1=(\lambda,1,1,\lambda^{-1})$ such that $\mathrm{rk}(T_1\lambda-1)=3$. Then

$$\mathbf{T}_1=\mathrm{MC}_{\lambda^{-1}}\circ\mathrm{MT}_{\boldsymbol{\Lambda}_1}(\mathbf{T})$$

is a rank three tuple. Taking $\boldsymbol{\Lambda}_2=(\lambda^{-1},-\lambda,1,-1)$ and $\boldsymbol{\Lambda}_3=(-1,\lambda^{-1},1,-\lambda)$ we obtain a rank two quadruple

$$\mathbf{S}=\mathrm{MT}_{\boldsymbol{\Lambda}_3}\circ\mathrm{MC}_{-\lambda}\circ\mathrm{MT}_{\boldsymbol{\Lambda}_2}(\mathbf{T}_1)$$

in $\mathrm{GO}_2(\mathbb{C})$ by Proposition 2.7. If $\mathbf{T}$ is quasi-unipotent the generated group is finite and therefore a pullback of a linearly rigid monodromy tuple of a Gauss hypergeometric differential equation by a well-known result of Klein (cf. Baldassarri and Dwork, 1979, Theorem 3.4). In any case a quadratic pullback yields a direct sum of two rank one tuples.
3. Taking $\Lambda^2$ in the case $P_3(4,8,8,10)$ we obtain a reducible tuple in $\mathrm{SO}_5(\mathbb{C})$ by the Scott formula. This excludes $\mathbf{J}(T_1)=\mathbf{J}(4)$ by Corollary A.3. Let $\boldsymbol{\Lambda}_1=(\lambda,1,1,\lambda^{-1})$ such that $\mathrm{rk}(T_1\lambda-1)=3$. Then

$$\mathbf{T}_1=\mathrm{MC}_{\lambda^{-1}}\circ\mathrm{MT}_{\boldsymbol{\Lambda}_1}(\mathbf{T})$$

is a rank four tuple. Taking $\boldsymbol{\Lambda}_2=(\lambda^{-1},-\lambda,1,-1)$ and $\boldsymbol{\Lambda}_3=(-1,\lambda^{-1},1,-\lambda)$ we obtain a rank four quadruple

$$\mathbf{S}=\mathrm{MT}_{\boldsymbol{\Lambda}_3}\circ\mathrm{MC}_{-\lambda}\circ\mathrm{MT}_{\boldsymbol{\Lambda}_2}(\mathbf{T}_1)$$

in $\mathrm{GO}_4(\mathbb{C})$ by Proposition 2.7. A quadratic pullback yields a five-tuple $\mathbf{T}_2$ with Jordan forms

$$\big(\big(\mathbf{J}(2),\mathbf{J}(2)\big),\big(\mathbf{J}(2),\mathbf{J}(2)\big),\big(\lambda,\lambda,\lambda^{-1},\lambda^{-1}\big),\big(\lambda,\lambda,\lambda^{-1},\lambda^{-1}\big),\big(\lambda_2^2,1,1,\lambda_2^{-2}\big)\big),$$

where $\mathrm{rk}(S_1-\lambda_2)=3$. Hence $\mathbf{T}_2$ can be written as a tensor product of two five-tuples $\mathbf{S}_1$ and $\mathbf{S}_2$ of rank two having two trivial entries. Since the $\mathbf{S}_i$ are linearly rigid the claim follows.
4. We can exclude the case $P_3(4,8,8,8)$. Since $\mathrm{MC}_{-1}(\mathbf{T})$ yields an orthogonal tuple of rank $m$, where $m=2+\mathrm{rk}(T_1-1)\in\{5,6\}$, we obtain an irreducible quadruple of rank four with three transvections, using the identities

$$\Lambda^2\big(\mathrm{Sp}_4(\mathbb{C})\big)=\mathrm{SO}_5(\mathbb{C}),\qquad \Lambda^2\big(\mathrm{SL}_4(\mathbb{C})\big)=\mathrm{SO}_6(\mathbb{C}).$$

But this contradicts the Scott formula. □

*3.3.2. The $P_4$ case*

**Theorem 3.14.** *In all the $P_4$ cases a symplectically rigid tuple $\mathbf{T}$ can be reduced via middle convolution operations and taking tensor products and rational pullbacks to a rank one tuple.*

**Proof.**

1. In the case $P_4(8,8,10,10)$ the dimension count contradicts the irreducibility.
2. A tuple $\mathbf{T}$ in the $P_4(6,8,10,10)$ case is linearly rigid.
3. In the case $P_4(6,6,10,10)$ the irreducibility of $\mathbf{T}$ implies that $\mathrm{rk}(T_4+1)=1$. Hence $\mathbf{S}=\mathrm{MC}_{-1}(\mathbf{T})$ is an orthogonal rank two tuple having two involutions. The claim follows as in the proof of Theorem 3.13.
4. A tuple $\mathbf{T}$ in the $P_4(8,8,8,10)$ case is linearly rigid.
5. In the case $P_4(6,8,8,10)$ the tuple $\mathbf{S}=\mathrm{MC}_{-1}(\mathbf{T})$ is an orthogonal tuple of rank five. A suitable sequence as in Proposition 2.7 yields an orthogonal tuple of rank two. The claim follows as in the proof of Theorem 3.13.

6. The case $P_4(6, 6, 8, 10)$ is excluded by the Scott formula.
7. In the case $P_4(8, 8, 8, 8)$ Scott's lemma shows that $\Lambda^2(\mathbf{T})$ has a three-dimensional orthogonal composition factor. By Corollary A.2 we get that $\mathbf{T}$ is a tensor product of two quadruples of rank two containing a trivial element. Hence we are in the linearly rigid case.
8. In the case $P_4(6, 8, 8, 8)$ we get that $\mathbf{S} = \Lambda^2(\mathbf{T})$ is reducible. The Scott formula and Corollary A.2 imply that $(S_1, S_2, -S_3, -S_4)$ splits into a trivial one-dimensional component and a four-dimensional one. Since the rank four tuple is linearly rigid the claim follows.
9. In the case $P_4(6, 6, 8, 8)$ $\mathrm{MC}_{-1}(\mathbf{T})$ is an orthogonal rank four tuple in $\mathrm{SO}_4(\mathbb{C})^4$, where $\mathbf{J}(T_3) = \mathbf{J}(T_4) = (\mathbf{J}(2), \mathbf{J}(2))$. Thus we can decompose it into a tensor product of two linearly rigid rank two tuples. $\square$

### 3.3.3. *The $P_5$ case*

**Theorem 3.15.** *In all $P_5$ cases a symplectically rigid tuple $\mathbf{T}$ can be reduced via middle convolution operations, taking tensor products and rational pullbacks to a rank one tuple.*

**Proof.**

1. In the case $P_5(10, 10, 10, 10, 10)$ the Scott formula implies that
$$\mathbf{J}(\mathbf{T}) = \big((\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), (-\mathbf{J}(2), -1, -1)\big).$$
Thus the tuple is linearly rigid, a so-called Jordan–Pochhammer tuple.
2. In the $P_5(8, 10, 10, 10, 10)$ case we get a contradiction to the Scott formula.
3. The $P_5(8, 8, 10, 10, 10)$ case is ruled out by the Scott formula.
4. In the case $P_5(8, 8, 8, 10, 10)$ the application of $\mathrm{MC}_{-1}$ yields an orthogonal rank four tuple with Jordan forms
$$\big((\mathbf{J}(2), \mathbf{J}(2)), (\mathbf{J}(2), \mathbf{J}(2)), (\mathbf{J}(2), \mathbf{J}(2)), (-1, 1, 1, 1), (-1, 1, 1, 1)\big).$$
Hence a quadratic pullback can be written as a tensor product of two linearly rigid six tuples of rank two with non-trivial Jordan forms $(\mathbf{J}(2), \mathbf{J}(2), -\mathbf{J}(2))$ each.
5. We can rule out the case $P_5(8, 8, 8, 8, 10)$. Otherwise $\mathbf{S} = \mathrm{MC}_{-1}(\mathbf{T})$ yields an orthogonal rank five tuple with Jordan forms $\mathbf{J}(S_1) = \cdots = \mathbf{J}(S_4) = (\mathbf{J}(2), \mathbf{J}(2), 1)$ and $\mathbf{J}(S_5) = (-1, -1, -1, -1, 1)$. Using $\Lambda^2\,\mathrm{Sp}_4 = \mathrm{SO}_5$ we get a symplectic rank four tuple with Jordan forms
$$\big((\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), (-1, -1, 1, 1)\big).$$
But this contradicts the Scott formula.
6. In the case $P_5(8, 8, 8, 8, 8)$ we apply $\mathrm{MC}_{-1}$ and obtain an orthogonal tuple $\mathbf{S}$ of rank six with Jordan forms $\mathbf{J}(S_1) = \cdots = \mathbf{J}(S_4) = -\mathbf{J}(S_5) = (\mathbf{J}(2), \mathbf{J}(2), 1, 1)$. Since $\Lambda^2\mathrm{SL}_4(\mathbb{C}) = \mathrm{SO}_6(\mathbb{C})$ we get a tuple of rank four with Jordan forms
$$\big((\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), (\mathbf{J}(2), 1, 1), \pm(i\mathbf{J}(2), i, i)\big).$$
The linear rigidity yields the claim. $\square$

**Remark 3.16.** In the $P_5(8, 8, 8, 8, 8)$ case the monodromy group $G = \langle \mathbf{T} \rangle$ is a finite two-group of order 32, where $Z(G) = G'$ and $G/G' \cong Z_2^4$.

## 4. Translation to differential operators

We translate the constructions for monodromy tuples used before to the level of differential operators in an appropriate way. Let as usual $\frac{d}{dz}$ be the derivation on $\mathbb{C}[z]$ defined by $\frac{d}{dz}(z) = 1$ and $\mathbb{C}[z, \partial] := \mathbb{C}[z][\partial]$ be the ring of differential operators with respect to $\frac{d}{dz}$. An element $L \in \mathbb{C}[z, \partial]$ with singular locus $S \subset \mathbb{C} \cup \{\infty\}$ can be regarded as a linear homogeneous differential equation on $\mathbb{P}^1 \setminus S$. Thus, we can investigate its induced local system $\mathbb{L}$ on $\mathbb{P}^1 \setminus S$ with respect to the following conventions.

**Conventions.** *We fix once and for all an orientation on $\mathbb{P}^1$ and denote the winding number of a closed path $\gamma$ around a point $p \in \mathbb{P}^1 \setminus \mathrm{im}(\gamma)$ by $\nu_\gamma(p)$. Furthermore, we denote the singular locus of a differential operator $L \in \mathbb{C}[z,\partial]$ by $S$. Having chosen an arbitrary base point $x_0 \in \mathbb{P}^1 \setminus S$, we attach to each $p \in \mathbb{P}^1$ a loop $\gamma_p$ starting at $x_0$ with $\nu_\gamma(p) = 1$ and $\nu_\gamma(s) = 0$ for all $s \in S \setminus \{p\}$. Then $\{\gamma_s\}_{s\in S}$ is a set of generators of $\pi_1(\mathbb{P}^1 \setminus S, x_0)$ and we equip $S$ with an ordering $S = \{s_1, \ldots, s_{r+1}\}$ such that their composition $\prod_{i=1}^{r+1} \gamma_{s_i}$ is homotopic to the trivial loop. We set the monodromy tuple associated to $L$ to be*

$$\mathbf{T} := (T_1, \ldots, T_{r+1}) := \big(\rho_{\mathbb{L}}(\gamma_{s_1}), \ldots, \rho_{\mathbb{L}}(\gamma_{s_{r+1}})\big) \in \mathrm{GL}(\mathbb{L}_{x_0})^{r+1}.$$

Mainly for computational and aesthetical reasons we use the so-called *logarithmic derivation* $z\frac{d}{dz}$ on $\mathbb{C}[z]$ and the ring of differential operators $\mathbb{C}[z,\vartheta] := \mathbb{C}[z][\vartheta]$ with respect to $z\frac{d}{dz}$, which can naturally be regarded as a subring of $\mathbb{C}[z,\partial]$. We call an operator $L = \sum_{i=0}^{n} a_i\vartheta^i$ with $a_i \in \mathbb{C}[z]$ *reduced*, if the greatest common divisor of all its coefficients $a_i$ is a unit. The *degree* $\deg(L)$ of $L$ is the maximal $i$ for which $a_i \neq 0$. Rearranging the coefficients, we also may write $L = \sum_{i=0}^{m} z^i P_i$, with $P_i \in \mathbb{C}[\vartheta]$. Recall, that $P_0$ is the *indicial equation* of $L$ at $z = 0$ and the roots of $P_0$ – considering $\vartheta$ as a formal variable – are the *exponents* $E$ of $L$. For each exponent $e \in E$, we have a formal solution $f \in z^\mu \mathbb{C}[[z]]^*$ of $L$ at $z = 0$, where $\mu \in (e + \mathbb{N}_0) \cap E$. We call $\mu$ the *exponent* of the solution $f$. The indicial equation and the exponents of $L$ at the other points $p \in \mathbb{P}^1$ can be obtained in the same way after having performed the transformation $z \mapsto z + p$ or $z \mapsto \frac{1}{z}$. We call $L$ *Fuchsian*, if the degree of its indicial equation at each point $p \in \mathbb{P}^1$ equals $\deg(L)$. This agrees with the usual definition of a Fuchsian operator as given in van der Put and Singer (2002, Section 6.2). As according to Deligne (1970) each operator of geometric origin has to be Fuchsian, we will perform all constructions with operators of this type.

Furthermore, let us briefly recall that there is a *universal Picard–Vessiot ring* $\mathcal{F}$ of $(\mathbb{C}[z], z\frac{d}{dz})$, i.e. for each $L \in \mathbb{C}[z,\vartheta]$ the set $\mathrm{Sol}_L := \{y \in \mathcal{F} \mid L(y) = 0\}$ can be regarded as a $\deg(L)$-dimensional $\mathbb{C}$-vector space. Therefore we call $\mathrm{Sol}_L$ the *solution space* of $L$. We will translate the operations used before on the level of monodromy tuples mainly via operations on the solutions of differential operators, which induce those monodromy tuples.

All local systems in the constructions done before are built up from local systems of the form

$$\Lambda_\alpha = \big(\alpha, 1, \alpha^{-1}\big)$$

and

$$\Lambda'_\alpha = \big(1, \alpha^{-1}, \alpha\big)$$

with respect to the points $\{0, 1, \infty\}$ for $a \in \mathbb{Q}$ and $\alpha = \exp(2\pi i a)$. Thus the basic operators we are dealing with are those of order one, which induce these monodromy tuples.

**Definition 4.1.** Let $a \in \mathbb{Q}$. We set

$$O_a := \vartheta - a \in \mathbb{C}[z,\vartheta]$$

and

$$I_a := \vartheta - z(\vartheta + a) \in \mathbb{C}[z,\vartheta].$$

**Remark 4.2.** The solution space of $O_a$ is spanned by the formal expression $z^a$, while the solution space of $I_a$ is spanned by the formal expression

$$\frac{1}{(1-z)^a}.$$

Both are algebraic over $\mathbb{Q}(z)$. Thus $O_a$ and $I_a$ are of geometric origin and induce precisely the monodromy tuples $\Lambda_\alpha$ and $\Lambda'_\alpha$. Two operators $O_a$ and $O_b$ induce the same monodromy tuple if and only if $a - b \in \mathbb{Z}$. The same statements hold for the operators $I_a$ and $I_b$.

### 4.1. Tensor product

We state the definition of the tensor product of differential operators as it is given in van der Put and Singer (2002, Chapter 2) and investigate some basic properties.

**Definition 4.3.** Let $L_1, L_2 \in \mathbb{C}[z,\vartheta]$ be reduced. The *tensor product* $L_1 \otimes L_2 \in \mathbb{C}[z,\vartheta]$ of $L_1$ and $L_2$ over $\mathbb{C}[z]$ is the reduced operator of minimal degree, whose solution space contains the set

$$\{y_1 y_2 \mid L_1(y_1) = L_2(y_2) = 0\} \subset \mathcal{F}.$$

**Remark 4.4.**

1. We always have $L_1 \otimes L_2 \in \mathbb{C}[z,\vartheta]$, as the vector space $V \subset \mathcal{F}$ spanned by $\{y_1 y_2 \mid L_1(y_1) = L_2(y_2) = 0\}$ is set-wise invariant under the natural action of the differential Galois group $G$ of $\mathcal{F} \supset \mathbb{C}[z]$. Thus by van der Put and Singer (2002, Lemma 2.17) the solution space of $L_1 \otimes L_2$ is exactly $V$.
2. We have $\deg(L_1 \otimes L_2) \leqslant \deg(L_1)\deg(L_2)$.
3. Symmetric and exterior powers of differential operators are defined similarly. For a reduced $L \in \mathbb{C}[z,\vartheta]$ we set $\mathrm{Sym}^n(L)$ to be the reduced operator of minimal degree whose solution space is spanned by the set
$$\{y_1 \cdot \cdots \cdot y_n \mid L(y_i) = 0 \text{ for all } i = 1,\dots,n\} \subset \mathcal{F}$$
and $\Lambda^n(L)$ to be the reduced operator of minimal degree whose solution space is spanned by the set
$$\{\mathrm{Wr}(y_1,\dots,y_n) \mid L(y_i) = 0 \text{ for all } i = 1,\dots,n\} \subset \mathcal{F},$$
where Wr denotes the Wronskian
$$\mathrm{Wr}(y_1,\dots,y_n) := \det\begin{pmatrix} y_1 & \cdots & y_n \\ \frac{d}{dz}y_1 & \cdots & \frac{d}{dz}y_n \\ \vdots & \vdots & \vdots \\ (\frac{d}{dz})^{n-1}y_1 & \cdots & (\frac{d}{dz})^{n-1}y_n \end{pmatrix}$$
with respect to the unique extension of $z\frac{d}{dz}$ to $\mathcal{F}$.

Since the solution space of $L_1 \otimes L_2$ is locally isomorphic to a subspace of the tensor product of the solution spaces of $L_1$ and $L_2$, we have the following

**Proposition 4.5.** *Let $L_1, L_2 \in \mathbb{C}[z,\vartheta]$ be irreducible with singular loci $S_1, S_2 \in \mathbb{C} \cup \{\infty\}$ and induced monodromy tuples $\mathbf{T}_1$ and $\mathbf{T}_2$ with respect to $x_0 \in \mathbb{P}^1 \setminus \{S_1 \cup S_2\}$. Then the following hold.*

1. *The monodromy tuple induced by $L_1 \otimes L_2$ is a direct summand of $\mathbf{T}_1 \otimes \mathbf{T}_2$.*
2. *The monodromy tuple induced by $\mathrm{Sym}^n L_1$ is a direct summand of $\mathrm{Sym}^n \mathbf{T}_1$.*
3. *The monodromy tuple induced by $\Lambda^n L_1$ is a direct summand of $\Lambda^n \mathbf{T}_1$.*

We especially get

**Corollary 4.6.** *Let $L \in \mathbb{C}[z,\vartheta]$ be a monic differential operator with induced monodromy tuple $\mathbf{T}$, $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha = \exp(2\pi i a)$. Then the monodromy tuple induced by $L \otimes I_a$ is precisely $\mathrm{MT}_{\Lambda'_\alpha}(\mathbf{T})$.*

### 4.2. Convolution and Hadamard product

In this section we investigate the Hadamard product with local systems of type $\Lambda'_\alpha$ using relations to the convolution with local systems of type $\Lambda_\beta$. Later on, we rather work with the Hadamard product than with the convolution on the level of differential operators, as the constructions we are after turn out to be easier then. We first define for $a \in \mathbb{Q} \setminus \mathbb{Z}$ the convolution of solutions of a Fuchsian operator with $z^a$ and the Hadamard product with $(1-z)^{-a}$, which span $\mathrm{Sol}_{O_a}$ and $\mathrm{Sol}_{I_a}$.

**Definition 4.7.** Let $L \in \mathbb{C}[z, \vartheta]$ be Fuchsian, $f$ a solution of $L$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$.

1. For two loops $\gamma_p, \gamma_q$ based at $x_0$ with $\nu_{\gamma_p}(q) = \nu_{\gamma_q}(p) = 0$ we define the *Pochhammer contour*
$$[\gamma_p, \gamma_q] := \gamma_p^{-1} \gamma_q^{-1} \gamma_p \gamma_q.$$
2. For $p \in \mathbb{P}^1$, the expression
$$C_a^p(f) := \int_{[\gamma_p, \gamma_z]} f(x)(z-x)^a \frac{dx}{z-x}$$
is called the *convolution* of $f$ and $z^a$ with respect to the Pochhammer contour $[\gamma_p, \gamma_z]$.
3. For $p \in \mathbb{P}^1$, the expression
$$H_a^p(f) := \int_{[\gamma_p, \gamma_z]} f(x)\left(1 - \frac{z}{x}\right)^{-a} \frac{dx}{x}$$
is called the *Hadamard product* of $f$ and $(1-z)^{-a}$ with respect to the Pochhammer contour $[\gamma_p, \gamma_z]$.

**Remark 4.8.**

1. In the sequel, we will frequently use the following formulae for integrals involving Pochhammer contours for $z \notin S$:

(a) $$\int_{\gamma_p \gamma_q} f(x)\,dx = \int_{\gamma_q} f(x)\,dx + \int_{\gamma_p} \rho_{\mathcal{L}}(\gamma_q)(f)(x)\,dx.$$

(b) $$\int_{[\gamma_p \gamma_q, \gamma_z]} f(x)(\lambda - x)^a \frac{dx}{\lambda - x} = C_a^q(f) + C_a^p\big(\rho_{\mathcal{L}}(\gamma_q)(f)\big).$$

(c) $$\int_{[\gamma_p^{-1}, \gamma_z]} f(x)(z-x)^a \frac{dx}{z-x} = -C_a^p\big(\rho_{\mathcal{L}}(\gamma_p)^{-1}(f)\big).$$

2. If $f \in (z-p)^{\mu}\mathbb{C}[[z-p]]$ near $z = p$, we get
$$C_a^p(f) = \big(1 - \exp(2\pi i \mu)\big) \int_{\gamma_z} f(x)(z-x)^a \frac{dx}{(z-x)} + \big(\exp(2\pi i a) - 1\big) \int_{\gamma_p} f(x)(z-x)^a \frac{dx}{(z-x)}.$$
In particular, we have
$$\int_{\gamma_z} f(x)(z-x)^a \frac{dx}{(z-x)} = \big(1 - \exp(2\pi i a)\big) \int_{x_0}^{z} f(x)(z-x)^a \frac{dx}{(z-x)}$$
and

$$\int_{\gamma_p} f(x)(z-x)^a \frac{dx}{(z-x)} = \big(1-\exp(2\pi i\mu)\big) \int_{x_0}^{p} f(x)(z-x)^a \frac{dx}{(z-x)},$$

if $\mu$ is not a negative integer. Thus we get

$$C_a^p(f) = \big(1-\exp(2\pi i\mu)\big)\big(1-\exp(2\pi ia)\big) \int_{p}^{z} f(x)(z-x)^a \frac{dx}{(z-x)}.$$

Note that the right hand side does not depend on the choice of the base point $x_0 \in \mathbb{P}^1 \setminus S$ and may be interpreted as a meromorphic function near $z = p$.

3. One checks that the convolution and the Hadamard product for a fixed Pochhammer contour $[\gamma_p, \gamma_z]$ are related by the following formulae:
   (a) $C_a^p(f) = (-1)^{a-1} H_{1-a}^p(z^a f)$.
   (b) $H_a^p(f) = (-1)^{-a} C_{1-a}^p(z^{a-1} f)$.

In order to find differential equations having solutions $C_a^p(f)$, we investigate some properties of the convolution.

**Lemma 4.9.** *Let $L \in \mathbb{C}[z,\vartheta]$ be Fuchsian, $f$ a solution of $L$, $a \in \mathbb{Q}\setminus\mathbb{Z}$, $p \in \mathbb{P}^1$ and $[\gamma_p, \gamma_z]$ a fixed Pochhammer contour. We have the following relations*:

1. $\frac{d}{dz} C_a^p(f) = C_a^p(\frac{d}{dz} f) = (a-1) C_{a-1}^p(f)$.
2. $C_a^p(zf) = z C_a^p(f) - C_{a+1}^p(f)$.
3. $C_a^p(z\frac{d}{dz} f) = (z\frac{d}{dz} - a) C_a^p(f)$.
4. $C_a^p(z^i f) = \prod_{j=0}^{i-1}(\frac{z\frac{d}{dz}}{a+j} - 1) C_{a+i}^p(f)$.

**Proof.** Using Leibniz's rule for differentiating under the integral sign we get

$$\begin{aligned}\frac{d}{dz} \int_{[\gamma_p,\gamma_z]} f(x)(z-x)^{a-1}\,dx &= \int_{[\gamma_p,\gamma_z]} f(x)\frac{d}{dz}(z-x)^{a-1}\,dx \\ &= -\int_{[\gamma_p,\gamma_z]} f(x)\frac{d}{dx}(z-x)^{a-1}\,dx.\end{aligned}$$

As the monodromy of $f(x)(z-x)^{a-1}$ along $[\gamma_p, \gamma_z]$ is trivial, integration by parts yields

$$-\int_{[\gamma_p,\gamma_z]} f(x)\frac{d}{dx}(z-x)^{a-1}\,dx = \int_{[\gamma_p,\gamma_z]} \left(\frac{d}{dx} f(x)\right)(z-x)^{a-1}\,dx$$

and hence the first result. The other statements can be obtained by direct computation and the results established before. □

Using those properties we get the following

**Proposition 4.10.** *Let $L = \sum_{i=0}^m z^i P_i(\vartheta) \in \mathbb{C}[z,\vartheta]$ be Fuchsian, $f$ a solution of $L$ and $a \in \mathbb{Q}\setminus\mathbb{Z}$. Then $C_a^p(f)$ is a solution of*

$$\mathcal{C}_a(L) := \sum_{i=0}^m z^i \prod_{j=0}^{i-1}(\vartheta + i - a - j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta - a)$$

*for each $p \in \mathbb{P}^1$.*

**Proof.** For $0 \leqslant i \leqslant m$ and $b \in \mathbb{Q} \setminus \mathbb{Z}$ we have

$$\begin{aligned} C^p_{b+i}(g) &= \frac{1}{\prod_{l=1}^{m-i}(b+m-l)} \big(C^p_{b+m}(g)\big)^{(m-i)} \\ &= z^{i-m} \frac{z^{m-i}}{\prod_{l=1}^{m-i}(b+m-l)} \big(C^p_{b+m}(g)\big)^{(m-i)} \\ &= z^{i-m} \frac{\prod_{k=0}^{m-i-1}(\vartheta - k)}{\prod_{l=1}^{m-i}(b+m-l)} C^p_{b+m}(g) \end{aligned}$$

for each $g$ which is a solution of some $R \in \mathbb{C}[z,\vartheta]$ by Lemma 4.9. Thus

$$\begin{aligned} 0 = C^p_b(Lf) &= \sum_{i=0}^m C^p_b\big(z^i P_i(\vartheta) f\big) = \sum_{i=0}^m \prod_{j=0}^{i-1}\left(\frac{\vartheta}{b+j} - 1\right) C^p_{b+i}\big(P_i(\vartheta) f\big) \\ &= \sum_{i=0}^m \prod_{j=0}^{i-1}\left(\frac{\vartheta}{b+j} - 1\right) z^{i-m} \frac{\prod_{k=0}^{m-i-1}(\vartheta-k)}{\prod_{l=1}^{m-i}(b+m-l)} C^p_{b+m}\big(P_i(\vartheta) f\big) \\ &= \sum_{i=0}^m z^{i-m} \prod_{j=0}^{i-1}\left(\frac{\vartheta+i-m}{b+j} - 1\right) \frac{\prod_{k=0}^{m-i-1}(\vartheta-k)}{\prod_{l=1}^{m-i}(b+m-l)} P_i\big(\vartheta-(b+m)\big) C^p_{b+m}(f) \\ &= \frac{\sum_{i=0}^m z^i \prod_{j=0}^{i-1}(\vartheta+i-m-b-j) \prod_{k=0}^{m-i-1}(\vartheta-k) P_i(\vartheta-(b+m))}{z^m \prod_{i=0}^{m-1}(b+i)} C^p_{b+m}(f). \end{aligned}$$

Setting $b = a - m$, we get the desired result. $\square$

An approach via so-called Euler-integrals can be found in Iwasaki et al. (1991, Chapter II.3) and yields a similar operator in $\mathbb{C}[z,\partial]$. We use the relations between the convolution and the Hadamard product to obtain an operator having solutions of the form $H^p_a(f)$.

**Corollary 4.11.** *Let $L = \sum_{i=0}^m z^i P_i \in \mathbb{C}[z,\vartheta]$ be Fuchsian, $f$ a solution of $L$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$. Then $H^p_a(f)$ is a solution of*

$$\mathcal{H}_a(L) := \sum_{i=0}^m z^i \prod_{j=0}^{i-1}(\vartheta + a + j) \prod_{k=0}^{m-i-1}(\vartheta - k) P_i$$

*for each $p \in \mathbb{P}^1$.*

Note that for an arbitrary Fuchsian operator $L$ the tuple $\mathrm{MC}_\alpha(\mathbf{T})$, resp. $\mathrm{MH}_{\alpha^{-1}}(\mathbf{T})$, is a subfactor of the monodromy tuple induced by $\mathcal{C}_a(L)$, resp. $\mathcal{H}_a(L)$. To induce the tuple $\mathrm{MC}_\alpha(\mathbf{T})$ we will restrict ourselves to operators, for which the expression $f(z)(y-z)^{a-1}$ is free of residues with respect to every $y \in \mathbb{P}^1$. This is guaranteed, if the operator $L$ is positive in the following sense.

**Definition 4.12.** Let $a \in \mathbb{Q} \setminus \mathbb{Z}$. A differential operator $L \in \mathbb{C}[z,\vartheta]$ is called *a-positive*, if $L$ is Fuchsian, has no exponents in $\mathbb{Z}_{<0}$ at each point $p \in \mathbb{C}$ and no exponents in $a + \mathbb{Z}_{<0}$ at $p = \infty$.

The next proposition justifies, that there is an operator in $\mathbb{C}[z,\vartheta]$, whose solution space is spanned by all $C^p_a(f)$, where $f$ is a solution of an $a$-positive operator $L$ and that this operator induces the desired monodromy tuple. As we have $C^p_a(f) = 0$ if $f$ is holomorphic at $p$ by Remark 4.8, we can concentrate ourselves on the expressions $C^s_a(f)$ for $s \in S$.

**Proposition 4.13.** *Let $a \in \mathbb{Q} \setminus \mathbb{Z}$, $L \in \mathbb{C}[z,\vartheta]$ be irreducible, a-positive with* $\deg(L) = n$, $S = \{s_1, \ldots, s_r, \infty\}$ *and $\alpha = \exp(2\pi i a)$. Let furthermore $\{f_1, \ldots, f_n\}$ be a basis of* $\mathrm{Sol}_L$,

$$R := \left(C_a^{s_1}(f_1), \ldots, C_a^{s_1}(f_n), \ldots, C_a^{s_r}(f_1), \ldots, C_a^{s_r}(f_n)\right)$$

*and*

$$V := \left\{R \cdot v \mid v \in \mathbb{C}^{nr}\right\}.$$

*Then the action of $C_\alpha(\mathbf{T})$ on $V$ as described in Section 2.2 is given by $\mathrm{MC}_\alpha(\mathbf{T})$.*

**Proof.** Due to Dettweiler and Reiter (2007, Section 4) the vector space $V$ is invariant under the action of the monodromy $C_\alpha(\mathbf{T})$. Let $F = (f_1, \ldots, f_n)$, $\mathcal{K}_k$ and $\mathcal{L}$ as in Section 2.2 and $v = (v_1, \ldots, v_r)^{tr}$, where $v_i \in \mathbb{C}^n$. Since $L$ is $a$-positive, $F \cdot v_k$ is holomorphic at $s_k$ for $v_k \in \ker(T_{s_k} - \mathrm{id})$ and we get $R \cdot v = 0$ for $v \in \mathcal{K}_k$. Thus we have

$$\dim_{\mathbb{C}}(V) \leqslant \sum_{s \in S \setminus \{\infty\}} \mathrm{rk}(T_s - \mathrm{id}).$$

We can choose for each $z \in \mathbb{P}^1 \setminus S$ a path $\gamma_z$ fulfilling our conventions such that

$$\gamma_{s_1} \cdots \gamma_{s_r} \gamma_z \gamma_\infty = 1.$$

With respect to the basis $\{f_1, \ldots, f_n\}$ of $\mathrm{Sol}_L$ and letting $C_a^p$ operate on each component of $F$, the elements of the monodromy group of $L$ operate via

$$C_a^p\left(\rho_{\mathbb{L}}(\gamma_{s_i})(F \cdot v)\right) = C_a^p(F) \cdot T_{s_i} v = C_a^p(F \cdot T_{s_i} v),$$

for each $v \in \mathbb{C}^n$ and each $0 \leqslant i \leqslant r$. Furthermore, by definition, the induced monodromy action of the path $\gamma_z$ on the integrand of $C_a^p(f)$ is just given by multiplication with $\alpha$. Using the rules established before, we have

$$\int\limits_{[\gamma_\infty^{-1}, \gamma_z]} F \cdot v(z - x)^a \frac{dx}{z - x} = \int\limits_{[\gamma_{s_1} \cdots \gamma_{s_r} \gamma_z, \gamma_z]} F \cdot v(z - x)^a \frac{dx}{z - x}$$
$$= \sum_{i=1}^{r} C_a^{s_i}(F) \cdot T_{s_{i+1}} \cdots T_{s_r} \alpha v$$

on the one hand and

$$\int\limits_{[\gamma_\infty^{-1}, \gamma_z]} F \cdot v(z - x)^a \frac{dx}{z - x} = -C_a^\infty(F) \cdot \alpha T_\infty^{-1} v = -C_a^\infty(F) \cdot \alpha T_{s_1} \cdots T_{s_r} v$$

on the other. Thus we get the relation

$$C_a^\infty(F) \cdot \alpha T_{s_1} \cdots T_{s_r} v = -\sum_{i=1}^{r} C_a^{s_i}(F) \cdot \alpha T_{s_{i+1}} \cdots T_{s_r} v$$

for each $v \in \mathbb{C}^n$. As the left hand side is zero for $v_\infty \in \ker(\alpha T_{s_1} \cdots T_{s_r} - \mathrm{id})$, rewriting the right hand side yields $R \cdot v = 0$ for each $v \in \mathcal{L}$.

Hence we get

$$\dim_{\mathbb{C}} V \leqslant \sum_{s \in S \setminus \{\infty\}} \mathrm{rk}(T_s - \mathrm{id}) - \left(n - \mathrm{rk}\left(\alpha T_\infty^{-1} - \mathrm{id}\right)\right).$$

By the definition of $\mathrm{MC}_\alpha(\mathbf{T})$ and comparing dimensions we get the result. □

**Remark 4.14.** With the notations used in the proposition above and by the relations between the convolution and the Hadamard product, assuming that $L \otimes O_{a-1}$ is $(1-a)$-positive and setting

$$\tilde{R} = \left(H_a^{s_1}(f_1), \ldots, H_a^{s_1}(f_n), \ldots, H_a^{s_r}(f_1), \ldots, H_a^{s_r}(f_n)\right)$$

and

$$\tilde{V} = \{\tilde{R} \cdot v \mid v \in \mathbb{C}^{nr}\},$$

the action of $H_{\alpha^{-1}}(\mathbf{T})$ on $\tilde{V}$ is given by $\mathrm{MH}_{\alpha^{-1}}(\mathbf{T})$.

As $C_\alpha(\mathbf{T})$ and $H_{\alpha^{-1}}(\mathbf{T})$ are induced by a Fuchsian systems, their Zariski closures over $\mathbb{C}$ are isomorphic to the differential Galois groups of the corresponding systems, see e.g. van der Put and Singer (2002, Corollary 5.2). By the preceding proposition and van der Put and Singer (2002, Lemma 2.17), there are non-trivial differential operators in $\mathbb{C}[z,\vartheta]$ whose solution spaces are exactly $V$, resp. $\tilde{V}$. This justifies the following definition.

**Definition 4.15.** Let $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $L \in \mathbb{C}[z,\vartheta]$ be irreducible.

1. If $L$ is $a$-positive, the *convolution* $L \star_C O_a$ of $L$ and $O_a$ is the non-trivial reduced operator of minimal degree in $\mathbb{C}[z,\vartheta]$ whose solution space contains the set
$$\bigcup_{p\in\mathbb{P}^1} \{C_a^p(f) \mid f \text{ is a solution of } L\}.$$
2. If $L \otimes O_{a-1}$ is $(1-a)$-positive, the *Hadamard product* $L \star_H I_a$ of $L$ and $I_a$ is the non-trivial reduced operator of minimal degree in $\mathbb{C}[z,\vartheta]$ whose solution space contains the set
$$\bigcup_{p\in\mathbb{P}^1} \{H_a^p(f) \mid f \text{ is a solution of } L\}.$$

As a consequence of Proposition 4.13, we get

**Corollary 4.16.** *Let $L \in \mathbb{C}[z,\vartheta]$ be irreducible with $\deg(L) = n$ and singular locus $S$. Let furthermore $S = \{0, s_2, \dots, s_r, \infty\}$, $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha = \exp(2\pi ia)$.*

1. *If $L$ is a-positive, $L \star_C O_a \in \mathbb{C}[z,\vartheta]$ is an irreducible Fuchsian right factor of $\mathcal{C}_a(L)$ of degree*
$$\deg(L \star_C O_a) = \sum_{s\in S\setminus\{\infty\}} \mathrm{rk}(T_s - \mathrm{id}) - \big(n - \mathrm{rk}\big(\alpha^{-1}T_\infty - \mathrm{id}\big)\big).$$
*Furthermore, its induced monodromy tuple is $\mathrm{MC}_\alpha(\mathbf{T})$.*
2. *If $L \otimes O_{a-1}$ is $(1-a)$-positive, $L \star_H I_a \in \mathbb{C}[z,\vartheta]$ is an irreducible Fuchsian right factor of $\mathcal{H}_a(L)$ of degree*
$$\deg(L \star_H I_a) = \sum_{s\in S\setminus\{0\}} \mathrm{rk}(T_s - \mathrm{id}) - \big(n - \mathrm{rk}(\alpha T_0 - \mathrm{id})\big).$$
*Furthermore, its induced monodromy tuple is $\mathrm{MH}_{\alpha^{-1}}(\mathbf{T})$.*

The degree of the operator $\mathcal{C}_a(L)$, resp. $\mathcal{H}_a(L)$, is possibly much higher than the degree of $L \star_C O_a$, resp. $L \star_H I_a$. As we know the degrees of $L \star_C O_a$, resp. $L \star_H I_a$, we can try to find those operators by a factorization of $\mathcal{C}_a(L)$, resp. $\mathcal{H}_a(L)$, into irreducible operators. Such a factorization is in general not unique, but yields a composition series of the solution space $W$ of the operator with respect to the action of its differential Galois group $G$, see e.g. Singer (1996, Proposition 2.11). It will turn out that in our cases we always have a factorization

$$\mathcal{H}_a(L) = \prod_{i=0}^{l} (\vartheta + c_i) R,$$

with $c_1, \dots, c_l \in \mathbb{C}$ and $\deg(R) = \deg(L \star_H I_a) > 1$. As then the only $\deg(L \star_H I_a)$-dimensional $G$-invariant subspace of $W$ on which $G$ acts irreducibly is exactly the solution space of $R$, we have $R = L \star_H I_a$. In particular, we have the following quite technical

**Proposition 4.17.** *Let $a \in \mathbb{Q} \setminus \mathbb{Z}$, $L = \sum_{i=0}^{m} z^i P_i \in \mathbb{C}[z, \vartheta]$ be irreducible and $\{0, \infty\} \subset S$. Let furthermore $k_0 \in \mathbb{N}$ maximal such that $\prod_{i=0}^{k_0-j-1}(\vartheta + a - 1 - i)$ divides $P_j$ for all $0 \leqslant j \leqslant k_0 - 1$ and $k_\infty \in \mathbb{N}$ maximal such that $\prod_{i=0}^{k_\infty-j-1}(\vartheta + 1 + i)$ divides $P_{m-j}$ for all $0 \leqslant j \leqslant k_\infty - 1$. Then:*

1. $\mathcal{H}_a(L) = \prod_{i=0}^{k_0-1}(\vartheta + a - 1 - i)\prod_{j=0}^{k_\infty-1}(\vartheta - m + 1 + j)R$, *with* $R \in \mathbb{C}[z, \vartheta]$.
2. *If $L \otimes O_{a-1}$ is $(1-a)$-positive, the operator $L \star_H I_a$ is an irreducible right factor of $R$.*
3. *If $L \otimes O_{a-1}$ is $(1-a)$-positive, $m = \sum_{s \in S \setminus \{0,\infty\}} \mathrm{rk}(T_s - \mathrm{id})$, $\mathrm{rk}(\exp(2\pi i a)T_0 - \mathrm{id}) = n - k_0$ and $\mathrm{rk}(T_\infty - \mathrm{id}) = n - k_\infty$, we have $R = L \star_H I_a$.*

**Proof.** By Corollary 4.11 we have

$$\mathcal{H}_a(L) = \sum_{i=0}^{m} z^i \prod_{j=0}^{i-1}(\vartheta + a + j) \prod_{k=0}^{m-i-1}(\vartheta - k)P_i.$$

Since $\mathcal{H}_a(L)$ has a left factor of the form $\vartheta + c$ with $c \in \mathbb{C}$ if and only if $\vartheta + c + i$ divides $\prod_{j=0}^{i-1}(\vartheta + a + j)\prod_{k=0}^{m-i-1}(\vartheta - k)P_i$ for each $0 \leqslant i \leqslant m$, we obtain the first part of the statement by a direct computation. The second part is a direct consequence of Corollary 4.16. To prove the third part, note that we have

$$\begin{aligned}\deg(R) &= n + m - k_0 - k_\infty \\ &= \sum_{s \in S\setminus\{0,\infty\}} \mathrm{rk}(T_s - \mathrm{id}) + \mathrm{rk}(T_\infty - \mathrm{id}) - \big(n - \mathrm{rk}(\alpha T_0 - \mathrm{id})\big) \\ &= \deg(L \star_H L_a)\end{aligned}$$

by Corollary 4.16. Now the action of the Galois group on the solution space as discussed above yields the result. □

A more general treatment of the factorization of $\mathcal{H}_a(L)$ will be discussed in a subsequent article.

**Example 4.18.** Let $a, b \in \mathbb{Q} \setminus \mathbb{Z}$. Recall that the monodromy tuple induced by $I_b$, where the singular locus of $I_b$ is extended by the apparent singularity $z = 0$, is given by $\mathbf{T} = (T_0, T_1, T_\infty) = (1, \beta^{-1}, \beta)$, where $\beta = \exp(2\pi i b)$. Thus we have $\deg(I_b \star_H I_a) = 2$ and

$$\mathcal{H}_b(I_a) = \vartheta^2 - z(\vartheta + b)(\vartheta + a) = I_b \star_H I_a.$$

Inductively, one shows that

$$I_{a_1} \star_H I_{a_2} \star_H \cdots \star_H I_{a_n} = \vartheta^n - z\prod_{i=1}^{n}(\vartheta + a_i).$$

In particular, each of those operators is of hypergeometric type.

The situation on local systems suggests, that the operation $\mathcal{H}_a$ is invertible. As we will see in the next lemma, this is not exactly the case.

**Lemma 4.19.** *Let $L = \sum_{i=0}^{m} z^i P_i \in \mathbb{C}[z, \vartheta]$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$. Then*

$$\mathcal{H}_{1-a}\big(\mathcal{H}_a(L) \otimes O_a\big) = \prod_{k=1}^{m-1}(\vartheta - k)\prod_{j=0}^{m-1}(\vartheta - a - j)(L \otimes O_a)\vartheta.$$

**Proof.** As we have

$$L \otimes O_a = \sum_{i=0}^{m} z^i P_i(\vartheta - a)$$

and

$$\mathcal{H}_a(L) \otimes O_a = \sum_{i=0}^{m} z^i \prod_{j=0}^{i-1}(\vartheta + j) \prod_{k=0}^{m-i-1} (\vartheta - a - k) P_i(\vartheta - a),$$

we obtain

$$\begin{aligned}\mathcal{H}_{1-a}\big(\mathcal{H}_a(L) \otimes O_a\big) &= \sum_{i=0}^{m} z^i \prod_{k=0}^{m-i-1} (\vartheta - k) \prod_{j=0}^{i-1}(\vartheta + j) \prod_{j=0}^{i-1}(\vartheta + 1 - a + j) \\ &\quad \prod_{k=0}^{m-i-1} (\vartheta - a - k) P_i(\vartheta - a) \\ &= \sum_{i=0}^{m} z^i \vartheta \prod_{k=1}^{m-1}(\vartheta - m + k + i) \prod_{j=0}^{m-1}(\vartheta - a - j + i) P_i(\vartheta - a)\end{aligned}$$

and hence the result. $\square$

Nevertheless, this lemma turns to be quite useful to determine solutions of $\mathcal{H}_a(L)$ involving logarithms as we will see in the next section.

## 5. Special solutions

The translation of the constructions appearing in Katz' algorithm to the level of differential operators enables us to compute certain local solutions of a differential operator produced by those constructions in an explicit way. To be more precise, given a Fuchsian operator $L$ which is constructed by tensor and Hadamard products of differential operators of lower order, we are sometimes able to state closed formulae for the coefficients of a local solution of the form $f = (z-p)^\mu \sum_{m=0}^\infty A_m (z-p)^m \in (z-p)^\mu \mathbb{C}[[z-p]]$ at $z = p \in \mathbb{C}$, resp. $f = t^\mu \sum_{m=0}^\infty A_m t^m \in t^\mu \mathbb{C}[[t]]$ for $t = \frac{1}{z}$. Those solutions will be called *special*. As stated in the preceding section, if $f = (z-p)^\mu \sum_{m=0}^\infty A_m (z-p)^m$ is a solution of the differential operator $L$ at $z = p$ and $g = (z-p)^\nu \sum_{m=0}^\infty B_m (z-p)^m$ is a solution of the differential operator $\tilde{L}$ at $z = p$, their Cauchy product

$$fg = (z-p)^{\mu+\nu} \sum_{m=0}^{\infty} \sum_{k=0}^{m} A_k B_{m-k} (z-p)^m$$

is a solution of $L \otimes \tilde{L}$ at $z = p$. Analogously, the self-Cauchy product $f^2$ is a solution of $\mathrm{Sym}^2 L$ at $z = p$ and setting $L = \tilde{L}$, the Wronskian

$$\mathrm{Wr}(f, g) = z(z-p)^{\nu+\mu-1} \sum_{m=0}^{\infty} \sum_{k=0}^{m} (2k + \mu - \nu - m) A_k B_{m-k} (z-p)^m$$

is a solution of $\Lambda^2 L$ at $z = p$. The situation turns out to be slightly more complicated for the middle Hadamard product $L \star_H I_a$ Classically one defines the Hadamard product $\star_H$ of two formal power series $\sum_{m=0}^\infty A_m z^m \in \mathbb{C}[[z]]$ and $\sum_{m=0}^\infty B_m z^m \in \mathbb{C}[[z]]$ by term-wise multiplication, i.e.

$$\sum_{m=0}^{\infty} A_m z^m \star_H \sum_{m=0}^{\infty} B_m z^m := \sum_{m=0}^{\infty} A_m B_m z^m.$$

As the terminology suggests, given a holomorphic solution $f = \sum_{m=0}^{\infty} A_m z^m$ of $L$ near $z = 0$, the expression

$$\sum_{m=0}^{\infty} (-1)^m \binom{-a}{m} A_m z^m$$

should be a solution of $L \star_H I_a$ near $z = 0$, as we have

$$(1-z)^{-a} = \sum_{m=0}^{\infty} (-1)^m \binom{-a}{m} z^m.$$

The following more general discussion will recover those solutions.

At $z = p$ the eigenfunctions of the local monodromy of a Fuchsian operator $L$ are elements of $(z-p)^{\mu}\mathbb{C}[[z-p]]^*$, where $\exp(2\pi i\mu)$ is the corresponding eigenvalue. For notational convenience, we use the following

**Conventions.** *Given $E \subset \mathbb{C}$ and two functions $f, g : E \to \mathbb{C}$ we write*

1. *$f \mathrel{\hat{=}} g$ if there is a $c \in \mathbb{C}^*$ such that $f(z) = cg(z)$ for all $z \in E$.*
2. *$\int_p^z f(x)\,dx$ for the integral of $f$ along the straight line $[0,1] \to E$, $t \mapsto (1-t)p + tz$, if it exists.*

*Furthermore, we will just write $\star$ instead of $\star_H$ in this section, which should lead to no confusion.*

The relation of $C_a^p(f)$ to the line integral given in Remark 4.8 yields the following

**Lemma 5.1.** *Let $f$ be an eigenfunction of the local monodromy of $L$ at $z = p \in \mathbb{C} \cup \{\infty\}$ and $\mu$ the exponent of $z^{a-1} f$ at $p$. Then we have*

$$H_a^p(f) \mathrel{\hat{=}} \begin{cases} \int_p^z x^{a-1} f(x)(z-x)^{-a}\,dx, & \mu \notin \mathbb{Z}, \\ 0, & \mu \in \mathbb{N}_0. \end{cases}$$

**Proof.** The statement follows directly from Remark 4.8. □

Recalling the well-known *Beta function*

$$\mathscr{B}(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

which is assumed to be the analytic continuation of the expression on the very right on $\mathbb{C} \setminus \{p+q \in \mathbb{Z}_{<0}\}$, a direct computation shows

**Lemma 5.2.**

1. *Let $f = z^{\mu} \sum_{m=0}^{\infty} A_m z^m$ be an eigenfunction of the local monodromy of $L$ at $z = 0$ with exponent $\mu \notin \mathbb{Z}_-$. Then*

$$C_a^0(f) \mathrel{\hat{=}} z^{\mu+a} \sum_{m=0}^{\infty} \mathscr{B}(\mu+1+m, a) A_m z^m.$$

2. *Let $t = \frac{1}{z}$ and $f = t^{\mu} \sum_{m=0}^{\infty} A_m t^m$ be an eigenfunction of the local monodromy of $L$ at $z = \infty$ with exponent $\mu \notin a + \mathbb{Z}_{<0}$. Then*

$$C_a^{\infty}(f) \mathrel{\hat{=}} t^{\mu-a} \sum_{m=0}^{\infty} \mathscr{B}(\mu-a+m, a) A_m t^m.$$

**Proof.**

1. By Remark 4.8, we have

$$C_a^0(f) \mathrel{\hat{=}} \int_0^z (z-x)^{a-1} f(x)\,dx = \sum_{m=0}^{\infty} A_m \int_0^z (z-x)^{a-1} x^{\mu+m}\,dx$$
$$= \sum_{m=0}^{\infty} A_m z^{\mu+m+a-1} \int_0^z \left(1-\frac{x}{z}\right)^{a-1} \left(\frac{x}{z}\right)^{\mu+m} dx$$
$$= \sum_{m=0}^{\infty} A_m z^{\mu+m+a} \int_0^1 (1-s)^{a-1} s^{\mu+m}\,ds$$

and thus the result.

2. We obtain the result similarly to the first part starting with

$$C_a^{\infty}(f) \mathrel{\hat{=}} \int_0^t x^{-1-a}(1-xz)^{a-1} f\left(\frac{1}{x}\right) dx. \qquad \square$$

Combining those statements yields

**Proposition 5.3.**

1. *Let $f$ be an eigenfunction of the local monodromy of $L$ at $z = p \in \mathbb{C}$. Let furthermore $z^{a-1} f = (z-p)^{\mu} \sum_{m=0}^{\infty} A_m (z-p)^m$. Then*

$$H_a^p(f) \mathrel{\hat{=}} \begin{cases} (z-p)^{\mu+1-a} \sum_{m=0}^{\infty} \mathscr{B}(\mu+1+m, 1-a) A_m (z-p)^m, & \mu \notin \mathbb{Z}, \\ 0, & \mu \in \mathbb{N}_0. \end{cases}$$

*In particular, if $L \otimes O_{a-1}$ is $(1-a)$-positive each $\mathbb{C}$-multiple of the right hand side is a solution of $L \star_H I_a$ near $z = p$.*

2. *Let $t = \frac{1}{z}$ and $f$ be an eigenfunction of the local monodromy of $L$ at $z = \infty$. Let furthermore $t^{1-a} f(t) = t^{\mu} \sum_{m=0}^{\infty} A_m t^m$. Then*

$$H_a^{\infty}(f) \mathrel{\hat{=}} \begin{cases} t^{\mu+a-1} \sum_{m=0}^{\infty} \mathscr{B}(\mu-1+a+m, 1-a) A_m t^m, & \mu \notin \mathbb{Z}, \\ 0, & \mu \in \mathbb{N}_0. \end{cases}$$

*In particular, if $L \otimes O_{a-1}$ is $(1-a)$-positive each $\mathbb{C}$-multiple of the right hand side is a solution of $L \star_H I_a$ near $z = \infty$.*

**Proof.** As seen before, we have

$$H_a^p(f) \mathrel{\hat{=}} \int_p^z x^{a-1} f(x)(z-x)^{-a}\,dx$$
$$\mathrel{\hat{=}} \int_0^{z-p} (x+p)^{a-1} f(x+p)(z-p-x)^{-a}\,dx$$

for $p \in \mathbb{C}$. Thus the result follows from Lemma 5.2 and Lemma 5.1. The case $p = \infty$ can be treated similarly. $\square$

**Remark 5.4.** If $f$ is holomorphic at $z=0$, one recovers the Hadamard product of formal power series mentioned in the introduction of the section. More generally, if $L$ has at $z=0$ a solution of the form $f=z^\nu \sum_{m=0}^\infty A_m z^m$ we get the solution

$$g=z^\nu \sum_{m=0}^\infty \mathscr{B}(\nu+a+m,1-a)A_m z^m = z^\nu \sum_{m=0}^\infty R(m)\mathscr{B}(a+m,1-a)A_m z^m$$

of $L \star_H I_a$ at $z=0$. Using Stirling's formula, one can show that $R(m)$ behaves asymptotically like $(\frac{\nu+a+m}{a+m})^{a-1}$.

Proposition 5.3 implies that each special solution $f$ of $L$ for which $z^{a-1}f$ is not a meromorphic eigenfunction near $z=p$ induces a special solution of $L \star_H I_a$, while the solutions $g$ for which $z^{a-1}g$ is holomorphic at $z=p$ do not contribute to the solution space of $L \star_H I_a$. Nevertheless, the following proposition asserts that solutions of the form $\ln g + r$ with $r \in \mathbb{C}[[z]]$ induce certain holomorphic solutions of $L \star_H I_a$.

**Proposition 5.5.**

1. *Let $L$ be irreducible and both functions*

$$z^{a-1}f=(z-p)^\mu \sum_{m=0}^\infty A_m(z-p)^m$$

*and $z^{a-1}g$ holomorphic at $z=p$. Let furthermore* ln *be a branch of the logarithm at $z=0$, $\ln(z-p)f+g$ a solution of $L$ at $z=p$ and $a\in\mathbb{Q}\setminus\mathbb{Z}$. Then*

$$H_a^p(f) \mathrel{\hat=} (z-p)^{\mu+1-a}\sum_{m=0}^\infty \mathscr{B}(\mu+1+m,1-a)A_m(z-p)^m.$$

2. *Let $L$ be irreducible $t=\frac{1}{z}$ and both functions*

$$t^{1-a}f=t^\mu \sum_{m=0}^\infty A_m t^m$$

*and $t^{1-a}g$ be holomorphic at $t=0$. Let furthermore* ln *be a branch of the logarithm at $t=0$, $\ln f+g$ a solution of $L$ at $t=0$ and $a\in\mathbb{Q}\setminus\mathbb{Z}$. Then*

$$H_a^\infty(f)=t^{\mu+a-1}\sum_{m=0}^\infty \mathscr{B}(\mu-1+a+m,1-a)A_m t^m.$$

**Proof.** Let $\tilde f = z^{a-1}f$ and $\tilde g = z^{a-1}g$. As the formal monodromy of $\ln(z-p)$ around $\gamma_p$ is given by $\ln(z-p)+2\pi i$, evaluating $H_a^p(\ln(z-p)f+g)$ yields

$$\begin{aligned}
H_a^p\big(\ln(z-p)f+g\big) &\mathrel{\hat=} C_{1-a}^p\big(\ln(z-p)\tilde f+\tilde g\big)\\
&= \int_{[\gamma_p,\gamma_z]} \ln(x-p)\tilde f(x)(z-x)^{-a}\,dx\\
&= -2\pi i\int_{\gamma_z} \tilde f(x)(z-x)^{-a}\,dx\\
&\quad + \big(\exp(-2\pi i a)-1\big)\int_{\gamma_p} \ln(x-p)\tilde f(x)(z-x)^{-a}\,dx
\end{aligned}$$

$$= -2\pi i\big(1-\exp(-2\pi i a)\big)\int_b^z \tilde f(x)(z-x)^{-a}\,dx$$
$$-2\pi i\big(\exp(-2\pi i a)-1\big)\int_b^p \tilde f(x)(z-x)^{-a}\,dx$$
$$= -2\pi i\big(1-\exp(-2\pi i a)\big)\int_p^z \tilde f(x)(z-x)^{-a}\,dx$$

and hence the result by Lemma 5.1 and Lemma 5.2. The second case can be treated analogously. □

Combining this result with Lemma 4.19, we get

**Lemma 5.6.** *Let $L \in \mathbb{C}[\vartheta, z]$, $a \in \mathbb{Q}\setminus\mathbb{Z}$, $p \notin \{0,\infty\}$, $f = (z-p)^{\mu}\sum_{m=0}^{\infty} A_m(z-p)^m \in \mathbb{C}[[z-p]]$, $r \in \mathbb{C}[[z-p]]$ and $\ln(z-p)f + r$ a solution of $\mathcal{H}_a(L)$ at $z=p$. Then*

1. $$h = z^{1-a}(z-p)^{a+\mu-1}\sum_{m=0}^{\infty}\mathscr{B}(\mu+1+m, a-1)A_m(z-p)^m$$
   *is a solution of $L$ at $z=p$.*
2. $H_a^p(h) \mathrel{\hat=} f$.

**Proof.** By Proposition 5.5 and Lemma 4.19, the expression

$$g = (z-p)^{\mu+a}\sum_{m=0}^{\infty}\mathscr{B}(\mu+1+m, a)A_m(z-p)^m$$

is a solution of $\prod_{k=1}^{m-1}(\vartheta-k)\prod_{j=0}^{m-1}(\vartheta-a-j)L^{z^{-a}}\vartheta$ at $z=p$.

As $p$ is no singularity of $\prod_{k=1}^{m-1}(\vartheta-k)\prod_{j=0}^{m-1}(\vartheta-a-j)$ and $\mu+a\notin\mathbb{Z}$, we have $L^{z^{-a}}\vartheta(g)=0$. Thus

$$z\frac{d}{dz}g = z(z-p)^{\mu+a-1}\sum_{m=0}^{\infty}\mathscr{B}(\mu+1+m, a-1)A_m z^m$$

is a solution of $L^{z^{-a}}$ and we obtain the first part of the statement. Setting $h = z^{1-a}\frac{d}{dz}g$ we get

$$z^{a-1}h = \frac{d}{dz}g = (z-p)^{a+\mu-1}\sum_{m=0}^{\infty}\mathscr{B}(\mu+1+m, a-1)A_m z^m.$$

Thus Proposition 5.3 yields

$$H_a^p(h) \mathrel{\hat=} (z-p)^{\mu}\sum_{m=0}^{\infty}\mathscr{B}(\mu+a+m, 1-a)\mathscr{B}(\mu+1+m, a-1)A_m(z-p)^m$$
$$\mathrel{\hat=} (z-p)^{\mu}\sum_{m=0}^{\infty}A_m(z-p)^m = f. \qquad \square$$

Rephrasing the lemma above, at a singular point $p \notin \{0,\infty\}$ the special holomorphic solutions $f$ are those, which induce solutions of the form $\ln(z-p)f + r$, where $r \in \mathbb{C}[[z-p]]$. In the geometric context solutions of this type turn out to be interesting as indicated in Candelas et al. (1998, Appendix B) and van Enckevort and van Straten (2004, Chapter 6).

## 6. Construction of Calabi–Yau operators

In this section, we combine the results of the preceding sections to construct families of irreducible Fuchsian differential operators inducing monodromy tuples of type $P_1$ and $P_2$. We will also compute special solutions of those operators at some of the singular points explicitly. Next, we investigate which of the operators constructed in the first part seem to be Calabi–Yau in the sense of Almkvist et al. (2010). As recently uncovered in Garbagnati and van Geemen (2010), unlike the definition of a Calabi–Yau operator given in Almkvist et al. (2010), there are families of Calabi–Yau threefolds, hence also Calabi–Yau operators, having no point of maximally unipotent monodromy. However, we restrict ourselves to the classical case of having such a point. In particular, the families $P_i$ for $i \geqslant 3$ cannot be induced by an operator corresponding to such a classical family. All operators we find using this method are covered by Almkvist et al. (2010, Appendix A), but in most of the cases we are unfortunately not able to show, whether the operators are Calabi–Yau.

In the sequel, we will use the notations introduced in the preceding sections and again let $t = \frac{1}{z}$ and abbreviate $\star_H$ by $\star$. As we have seen before, the construction of monodromy tuples of type $P_1$ and $P_2$ splits into four cases, each of which we will cover by the subsequent theorems. Furthermore, we only construct those operators $L$ for which zero is the only exponent at $z = 0$ and choose the singular locus of $L$ to be $S = \{0, 1, \infty\}$. We collect the remaining exponents $\lambda_{1,1}, \ldots, \lambda_{4,1}$ at $z = 1$ and $\lambda_{1,\infty}, \ldots, \lambda_{4,\infty}$ at $z = \infty$ in its Riemann scheme

$$\mathcal{R}(L) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & \lambda_{1,1} & \lambda_{1,\infty} \\ 0 & \lambda_{2,1} & \lambda_{2,\infty} \\ 0 & \lambda_{3,1} & \lambda_{3,\infty} \\ 0 & \lambda_{4,1} & \lambda_{4,\infty} \end{array} \right\}.$$

In all occurring cases, the Jordan forms of the local monodromies can be read off directly from the Riemann scheme, as only repeated exponents turn out to induce logarithms. Proofs of those statements which can be obtained directly using the methods established before are omitted. For the sake of clarity, we frequently use well-known hypergeometric identities as stated in Bailey (1935) without any further comment. Furthermore, to avoid an even larger zoo of brackets we write $L \star I_a \star I_b$ instead of $(L \star I_a) \star I_b$ and $L \star I_a \otimes O_b$ instead of $(L \star I_a) \otimes O_b$ for $L \in \mathbb{C}[\vartheta, z]$ and $a, b \in \mathbb{Q} \setminus \mathbb{Z}$.

**Theorem 6.1** *(The $P_1(4, 10, 4)$ case). Let $a, b \in \mathbb{Q} \setminus \mathbb{Z}$. A two parameter family of operators inducing monodromy tuples of type $P_1(4, 10, 4)$ is given by*

$$\begin{aligned} \mathcal{P}_1^{(a,b)}(4, 10, 4) &:= I_a \star I_{1-a} \star I_b \star I_{1-b} \\ &= \vartheta^4 - z(\vartheta + a)(\vartheta + 1 - a)(\vartheta + b)(\vartheta + 1 - b). \end{aligned}$$

*The Riemann scheme reads*

$$\mathcal{R}\big(\mathcal{P}_1^{(a,b)}(4, 10, 4)\big) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a \\ 0 & 1 & 1-a \\ 0 & 1 & b \\ 0 & 2 & 1-b \end{array} \right\}.$$

*Special solutions of this operator are $f = \sum_{m=0}^{\infty} A_m z^m$ at $z = 0$, where*

$$A_m = \binom{a+m-1}{m}\binom{m-a}{m}\binom{b+m-1}{m}\binom{m-b}{m},$$

*$g=(z-1)\sum_{m=0}^{\infty}B_m(z-1)^m$ at $z=1$, where*

$$B_m=\frac{b-1}{m+1}\binom{1+m-b}{m}\sum_{l=0}^{m}(-1)^l\binom{-b}{m-l}\frac{{}_3F_2\left(\begin{smallmatrix}a,\ -l,\ 1-a\\ 1,\ b-l\end{smallmatrix}\,\middle|\,1\right)}{b-l-1}$$

*and $h_\gamma=t^\gamma\sum_{m=0}^{\infty}C_m^{(\gamma)}t^m$ at $z=\infty$, where $\gamma\in E=\{a,1-a,b,1-b\}$ and*

$$C_m^{(\gamma)}=\mathscr{B}(\gamma+m,1-a)\mathscr{B}(\gamma+m,a)\mathscr{B}(\gamma+m,1-b)\mathscr{B}(\gamma+m,b).$$

*Moreover, $g$ is the conifold-period of $\mathcal{P}_1^{(a,b)}(4,10,4)$ at $z=1$, i.e. there is an $r\in(z-1)\mathbb{C}[[z-1]]$ such that $\ln(z-1)g+r$ is a solution of $\mathcal{P}_1^{(a,b)}(4,10,4)$ at $z=1$.*

**Proof.** It is clear that $I_a\star I_{1-a}\star I_b\star I_{1-b}$ induces a monodromy tuple of type $P_1(4,10,4)$. As in Example 4.18, we get

$$\begin{aligned}I_a\star I_{1-a}\star I_b\star I_{1-b}&=\mathcal{H}_{1-b}\big(\mathcal{H}_b\big(\mathcal{H}_{1-a}(I_a)\big)\big)\\&=\vartheta^4-z(\vartheta+a)(\vartheta+1-a)(\vartheta+b)(\vartheta+1-b).\end{aligned}$$

The formulae for $A_m$, $B_m$ and $C_m^{(\gamma)}$ can be obtained directly using Proposition 5.5 and exchanging the roles of $a$, $1-a$, $b$ and $1-b$ freely. It remains to show, that $g$ is the conifold-period at $z=1$. As $e=1$ is an exponent of multiplicity two at $z=1$, the method of Frobenius yields a solution $\ln(z-1)\tilde{g}+r$ of $\mathcal{P}_1^{(a,b)}(4,10,4)$ at $z=1$, where $\tilde{g}\in(z-1)\mathbb{C}[[z-1]]$ and $r\in(z-1)\mathbb{C}[[z-1]]$. Applying the first statement of Lemma 5.6 yields a solution $\omega\in(z-1)^{1-b}\mathbb{C}[[z-1]]$ of $I_a\star I_{1-a}\star I_b$. As $1-b$ is the only exponent of $I_a\star I_{1-a}\star I_b$ at $z=1$ lying in $-b+\mathbb{Z}$, we have $\omega\hat{=}H_b^1(H_{1-a}^1((1-z)^{-a}))$. Applying the second statement of Lemma 5.6 yields

$$\tilde{g}\hat{=}H_{1-b}^1(\omega)\hat{=}H_{1-b}^1\big(H_b^1\big(H_{1-a}^1\big((1-z)^{-a}\big)\big)\big)\hat{=}g.\qquad\square$$

**Theorem 6.2** *(The $P_1(4,8,4)$ case). Let $a\in\mathbb{Q}\setminus(\frac{1}{4}+\mathbb{Z}\cup\frac{3}{4}+\mathbb{Z})$ and $b\in\mathbb{Q}\setminus(\frac{1}{4}+\mathbb{Z}\cup\frac{3}{4}+\mathbb{Z})$. A two parameter family of operators inducing monodromy tuples of type $P_1(4,4,8)$ is given by*

$$\begin{aligned}\mathcal{P}_1^{(a,b)}(4,8,4)&:=\big(\Lambda^2\big((I_{\frac{1}{4}+a}\star I_{\frac{1}{4}-a}\otimes O_{-\frac{1}{2}})\star I_{\frac{3}{4}+b}\star I_{\frac{3}{4}-b}\big)\otimes O_{\frac{3}{2}}\big)\star I_{\frac{3}{2}}\\&=64\vartheta^4+z\big(-128\vartheta^4-256\vartheta^3+\vartheta^2\big(128\big(a^2+b^2\big)-304\big)\big)\\&\quad+z\big(\vartheta\big(128\big(a^2+b^2\big)-176\big)+48\big(a^2+b^2\big)+256a^2b^2-39\big)\\&\quad+64z^2(\vartheta+1-a-b)(\vartheta+1+a-b)(\vartheta+1-a+b)(\vartheta+1+a+b).\end{aligned}$$

*The Riemann scheme reads*

$$\mathcal{R}\big(\mathcal{P}_1^{(a,b)}(4,8,4)\big)=\left\{\begin{array}{ccc}0&1&\infty\\\hline 0&-\frac{1}{2}&1-a-b\\0&0&1+a-b\\0&1&1-a+b\\0&\frac{3}{2}&1+a+b\end{array}\right\}.$$

*Special solutions of this operator are given by $f=\sum_{m=0}^{\infty}A_mz^m$ at $z=0$ with*

$$A_m=\binom{\frac{1}{2}+m}{m}\sum_{k=0}^{m}\left(2k-\frac{1}{2}-m\right)\alpha\left(-\frac{1}{2},k\right)\alpha(0,m-k),$$

*where*

$$\alpha(\nu,m) := \mathscr{B}\left(\frac{3}{4}+a+\nu+m, \frac{1}{4}-a\right)\mathscr{B}\left(\frac{3}{4}-a+\nu+m, \frac{1}{4}+a\right) \times \mathscr{B}\left(\frac{3}{4}+b+\nu+m, \frac{3}{4}-b\right)\mathscr{B}\left(\frac{3}{4}-b+\nu+m, \frac{3}{4}+b\right)$$

*and* $h_{(\mu,\nu)} = t^{\mu+\nu}\sum_{m=0}^{\infty} C_m^{(\mu,\nu)} t^m$ *at* $z=\infty$, *where*

$$C_m^{(\mu,\nu)} = \mathscr{B}\left(\nu+\mu+m, -\frac{1}{2}\right)\sum_{k=0}^{m}(2k+\mu-\nu-m)\delta(\mu,k)\delta(\nu,m-k),$$

*with*

$$\delta(\mu,k) := \mathscr{B}\left(\mu-\frac{1}{4}+k, \frac{3}{4}-a\right)\mathscr{B}\left(\mu-\frac{1}{4}+k, \frac{3}{4}+a\right) \times \mathscr{B}\left(\mu+\frac{1}{4}+k, \frac{1}{4}-b\right)\mathscr{B}\left(\mu+\frac{1}{4}+k, \frac{1}{4}+b\right),$$

*for* $\mu \in \{\frac{1}{2}+a, \frac{1}{2}-a\}$ *and* $\nu \in \{\frac{1}{2}+b, \frac{1}{2}-b\}$.

**Theorem 6.3** *(The* $P_2(4,6,6)$ *case). Let* $a,b \in \mathbb{Q}\setminus\mathbb{Z}$. *A two parameter family of operators inducing monodromy tuples of type* $P_2(4,6,6)$ *is given by*

$$\begin{aligned}\mathcal{P}_2^{(a,b)}(4,6,6) &:= \mathrm{Sym}^2(I_a \star I_b \otimes I_{\frac{1-a-b}{2}}) \star I_{\frac{1}{2}}\\ &= 4\vartheta^4 - 2z(2\vartheta+1)^2\big(\vartheta^2+\vartheta+2ab-a+1-b\big)\\ &\quad - z^2(2\vartheta+3)(2\vartheta+1)(b-1-a-\vartheta)(b+1-a+\vartheta).\end{aligned}$$

*The Riemann scheme reads*

$$\mathcal{R}\big(\mathcal{P}_2^{(a,b)}(4,6,6)\big) = \left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & -\frac{1}{2}+a+b & 1+a-b \\ 0 & \frac{3}{2}-a-b & 1-a+b \end{array}\right\}.$$

*Special solutions of this operator are* $f = \sum_{m=0}^{\infty} A_m z^m$ *at* $z=0$, *where*

$$A_m = \binom{m-\frac{1}{2}}{m}\sum_{k=0}^{m}\binom{a+k-1}{k}\binom{b+k-1}{k}\binom{m-k-a}{m-k}\binom{m-k-b}{m-k},$$

$g_{(a,b)}$ *and* $g_{(1-a,1-b)}$ *at* $z=1$, *where* $g_{(a,b)} = (z-1)^{\frac{3}{2}-a-b}\sum_{m=0}^{\infty} B_m^{(a,b)}(z-1)^m$, *with*

$$B_m^{(a,b)} = \mathscr{B}\left(2-a-b+m, \frac{1}{2}\right)\sum_{l=0}^{m}\mathscr{B}(1-b+l, 1-a)\alpha(l)\binom{-\frac{1}{2}}{m-l}\binom{a-1}{l}$$

*and*

$$\alpha(l) = {}_4F_3\left(\left.\begin{array}{c}-l,\ 1-b,\ 1-a,\ a-1-l+b\\ b-l,\ a-l,\ 2-a-b\end{array}\right|1\right)$$

*and* $h_{(a,b)}$ *and* $h_{(1-a,1-b)}$ *at* $z=\infty$, *where* $h_{(a,b)} = t^{1-a+b}\sum_{m=0}^{\infty} C_m^{(a,b)} t^m$ *with*

$$C_m^{(a,b)} = \mathscr{B}\left(1-a+b+m, \frac{1}{2}\right)\sum_{l=0}^{m}\delta(l)\delta(m-l)$$

*and*

$$\delta(l) = {}_3F_2\left(\begin{matrix} b,\ b,\ -l \\ \frac{1}{2}-a+b,\ \frac{1}{2}(1+a+b)-l \end{matrix}\,\middle|\, 1\right)\binom{-\frac{1}{2}(1+a+b)+l}{l}.$$

**Theorem 6.4** *(The $P_2(4,6,8)$ case). Let $a, b \in \mathbb{Q}\setminus\mathbb{Z}$. A two parameter family of operators inducing monodromy tuples of type $P_2(4,6,8)$ is given by*

$$\begin{aligned}\mathcal{P}_2^{(a,b)}(4,6,8) &:= (I_a \star I_a \otimes I_{1-a}) \star I_b \star I_{1-b} \\ &= \vartheta^4 - z(\vartheta+b)(\vartheta+1-b)\big(2\vartheta^2+2\vartheta+a^2-a+1\big) \\ &\quad + z^2(\vartheta+b)(\vartheta+1-b)(\vartheta+b+1)(\vartheta+2-b).\end{aligned}$$

*The Riemann scheme reads*

$$\mathcal{R}\big(\mathcal{P}_2^{(a,b)}(4,6,8)\big) = \left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & b \\ 0 & 1 & 1-b \\ 0 & a & 1+b \\ 0 & 1-a & 2-b \end{array}\right\}.$$

*Special solutions of this operator are $f = \sum_{m=0}^{\infty} A_m z^m$ at $z=0$, where*

$$A_m = \binom{b+m-1}{m}\binom{m-b}{m}\sum_{k=0}^{m}\binom{a+m-k-1}{m-k}^2\binom{k-a}{k}$$

*and $g_\gamma = (z-1)^\gamma \sum_{m=0}^{\infty} B_m^{(\gamma)}(z-1)^m$ at $z=1$, where*

$$B_m^{(\gamma)} = \mathscr{B}(1+\gamma-b+m, b)\sum_{l=0}^{m}(-1)^l\alpha(l)\mathscr{B}(1-b+l,-\gamma)\binom{-b}{m-l}\binom{l-1+\gamma}{\gamma-1},$$

*with $\gamma \in \{a, 1-a\}$, where*

$$\alpha(l) = {}_3F_2\left(\begin{matrix} -l,\ \gamma,\ \gamma \\ 1+\gamma,\ b-l \end{matrix}\,\middle|\, 1\right).$$

Now we investigate which of the operators constructed before are differential Calabi–Yau operators in the spirit of Almkvist et al. (2010). We first recall the definition of those objects, which still is quite conjectural and state some of their basic properties. From the geometric point of view, the solutions of a Calabi–Yau operator of order $n$ should correspond to periods of a family of Calabi–Yau manifolds of dimension $n-1$ with Picard number one. In this sense, Calabi–Yau operators should be special Picard–Fuchs operators, which can't be defined from the differential algebraic point of view in a proper way yet. According to our definition, Calabi–Yau operators respect common conjectures for a differential operator to be Picard–Fuchs, see e.g. Kontsevich and Zagier (2001). Some of the arithmetic conditions for a differential operator to be Calabi–Yau are basically motivated by approaches of mirror symmetry as discussed in Candelas et al. (1998), but still seem to be quite mysterious.

**Definition 6.5.** For $n \geqslant 2$, an irreducible operator $L = \partial^n + \sum_{i=0}^{n-1} a_i\partial^i \in \mathbb{Q}(z)[\partial]$ is called *Calabi–Yau operator* if it satisfies the following conditions.

(CY-1) The point $z=0$ is a regular singularity of $L$ and zero is the only exponent at this point.
(CY-2) $L$ has a solution $y_0$ which is $N$-integral at $z=0$, i.e. at $z=0$ it is of the form

$$y_0 = 1 + \sum_{m=1}^{\infty} A_m z^m \in \mathbb{Q}[[z]],$$

with $N^m A_m \in \mathbb{Z}$ for each $m \geqslant 1$ and a fixed $N \in \mathbb{N}$.
(CY-3) We have

$$L\alpha = \alpha L^*$$

for a non-trivial solution $\alpha$ of the differential equation $\omega' = -\frac{2}{n} a_{n-1}\omega$. Here

$$L^* = \partial^n + \sum_{i=0}^{n-1} (-1)^{n+i} \partial^i a_i \in \mathbb{C}(z)[\partial]$$

denotes the dual operator of $L$.
(CY-4) There is a solution $y_1$ linearly independent of $y_0$ given in (CY-3), such that the differential equation

$$\omega' = \left(\frac{y_1}{y_0}\right)' \omega$$

has a non-trivial solution $q \in z + z^2\mathbb{Q}[[z]]$ at $z=0$ which is $N$-integral. Such a solution is often called the *q-coordinate* or special coordinate of $L$ at $z=0$.

**Remark 6.6.**

1. The definition stated above should rather be seen as a preliminary list of properties than a rigorous description. One could also claim further integrality conditions, like the integrality of the so-called Yukawa coupling and instanton numbers, see e.g. Almkvist et al. (2010) and Almkvist and Zudilin (2005).
2. If $L$ is a Calabi–Yau operator of order two, one can show that $\mathrm{Sym}^3(L)$ is a Calabi–Yau operator of order four. We do not collect operators of this type.

By the construction done in Theorems 6.1–6.3, we get

**Lemma 6.7.** *Each of the operators* $\mathcal{P}_1^{(a,b)}(4,10,4)$, $\mathcal{P}_1^{(a,b)}(4,8,4)$, $\mathcal{P}_2^{(a,b)}(4,6,8)$ *and* $\mathcal{P}_2^{(a,b)}(4,6,6)$ *constructed in Theorem* 6.1–6.3 *fulfills the properties* (CY-1)–(CY-3).

**Proof.** Property (CY-1) can be read off the corresponding Riemann scheme directly. Using Dwork et al. (1994, Theorem I.4.3 and Formula II.4.6), one shows that the unique solution at $z=0$ lying in $1+\mathbb{Q}[[z]]$ of each operator is $N$-integral. Finally, condition (CY-3) can be obtained by a direct computation. □

It remains to investigate which of the operators fulfill property (CY-4). Although there have recently been many improvements in the technique of showing this property, see e.g. Krattenthaler and Rivoal (2010, 2011), we are in most of the cases not able to decide whether condition (CY-4) holds or not. Let us point out that for each operator constructed here it is also possible to compute a solution of the form $\ln(z)y_0 + y_1$ by taking $\frac{d}{d\mu} y_0|_{\mu=0}$ of the holomorphic solution $y_0 = \sum_{\nu=\mu}^{\infty} f(\mu)z^{\mu}|_{\mu=0}$ as it is described in Ince (1956, Chapter 16) but that we are often not able to check, whether the criterion (Krattenthaler and Rivoal, 2010, Proposition 4.1) holds or the series can be treated by a specialization of Krattenthaler and Rivoal (2011, Theorem 2). Thus we only checked numerically, if property (CY-4) is fulfilled or not. From a geometric point of view, it would be natural that the elements of the monodromy tuple lie – up to simultaneous conjugation – in $\mathrm{Sp}_4(\mathbb{Z})$ in order to have an integral structure on the limiting mixed Hodge structure for the corresponding family. Indeed, for operators of families $\mathcal{P}_1^{(a,b)}(4,10,4)$, $\mathcal{P}_2^{(a,b)}(4,6,8)$ and $\mathcal{P}_2^{(a,b)}(4,6,6)$, the $q$-coordinate seems not to be $N$-integral, if this is not the case. For family $\mathcal{P}_1^{(a,b)}(4,8,4)$, the second exterior power is of hypergeometric type and can hence be treated analogously. Furthermore, if we alter the exponents by integers, we get an operator which is of the same type. In the geometric situation, the cyclic vector corresponding to the Calabi–Yau operator is a holomorphic $n$-form and hence the cyclic vector for a monic operator of the same type corresponds to a mixed form, unless we just multiplied it with a function, which is holomorphic near $z=0$. Thus it is plausible, that two monic Calabi–Yau operators are up to conjugation by an algebraic function never of the same type, although this question is open. This is also reflected in our observations, which lead to the following

**Conjecture.** *An* $\mathrm{Sp}_4(\mathbb{C})$*-rigid tuple consisting of quasi-unipotent elements and having a maximally unipotent element is induced by a differential Calabi–Yau operator if and only if the elements of its second exterior power lie up to simultaneous conjugation in* $\mathrm{SO}_5(\mathbb{Z})$*. Furthermore, the inducing operator is unique.*

In the sequel we state which of the cases in each of the families correspond to operators listed in Almkvist et al. (2010, Appendix A) and refer to the number of the operator stated there. Note that the operators constructed here have singular locus $\{0,1,\infty\}$, so we get the corresponding operators after having performed a transformation of the form $z \mapsto \lambda z$ with $\lambda \in \mathbb{Q}^*$, which leaves the properties (CY-1)–(CY-4) untouched and changes the singular locus to $\{0, \frac{1}{\lambda}, \infty\}$. It is remarkable that after having performed the transformation the coefficients of the $q$-coordinate are minimal over $\mathbb{Z}$, meaning that they are all lying in $\mathbb{Z}$ and there is no $\alpha \in \mathbb{Z}$ such that $\alpha^m$ divides the $m$-th coefficient for each $m \in \mathbb{N}$. Furthermore for each series of operators the transformation can be done uniformly. Let therefore in the sequel for $a = \frac{r}{s}$, where $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ are coprime,

$$\beta : \mathbb{Q} \setminus \{0\} \to \bar{\mathbb{Z}}, \quad a \mapsto s \prod_{i=1}^{n} s_i^{\frac{1}{s_i - 1}},$$

where $s_1, \ldots, s_n$ denote the distinct prime divisors of $s$.

(i) The $P_1(4,10,4)$ case:
Having performed the transformation $z \mapsto \beta(a)^2 \beta(b)^2 z$, we get the following Calabi–Yau operators

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{5}$ | $\frac{1}{8}$ | $\frac{1}{10}$ | $\frac{1}{12}$ |
| $b$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{5}$ | $\frac{3}{8}$ | $\frac{3}{10}$ | $\frac{5}{12}$ |
| Nr. | 3 | 5 | 6 | 14 | 4 | 11 | 8 | 10 | 12 | 13 | 1 | 7 | 2 | 9 |

(ii) The $P_1(4,8,4)$ case:
To make our observations more transparent, we substitute $c = 2a + \frac{1}{2}$ and $d = 2b + \frac{1}{2}$. Having performed the transformation $z \mapsto 4\beta(c)^2\beta(d)^2 z$, we get the following Calabi–Yau operators

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{5}$ | $\frac{1}{8}$ | $\frac{1}{10}$ | $\frac{1}{12}$ |
| $d$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{5}$ | $\frac{3}{8}$ | $\frac{3}{10}$ | $\frac{5}{12}$ |
| Nr. | $\tilde{3}$ | $\tilde{5}$ | $\tilde{6}$ | $\tilde{14}$ | $\tilde{4}$ | $\tilde{11}$ | $\tilde{8}$ | $\tilde{10}$ | $\tilde{12}$ | $\tilde{13}$ | $\tilde{1}$ | $\tilde{7}$ | $\tilde{2}$ | $\tilde{9}$ |

where the number $\tilde{i}$ refers to the operators defined in Almkvist (2006). As shown there, those operators are equivalent to 206–219 in Almkvist et al. (2010). This family contains elements whose induced monodromy group is not a subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. Note that the operator

$$Q_2^{c,d} := \left(\Lambda^2 \mathcal{P}_1^{c,d}(4,8,8)\right) \otimes O_1 \otimes I_{-\frac{3}{2}}$$

is of hypergeometric type. Its Riemann scheme reads

$$\mathcal{R}\big(Q_2^{c,d}\big) = \left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & \frac{1}{2} \\ 0 & 1 & c \\ 0 & \frac{3}{2} & d \\ 0 & 2 & 1-c \\ 0 & 3 & 1-d \end{array}\right\}.$$

For the values of $c$ and $d$ in the table above, the monodromy of $Q_2^{c,d}$ can be realized in $\mathrm{SO}_5(\mathbb{Z})$.

(iii) The $P_2(4,6,6)$ case:
Having performed the transformation $z \mapsto 4\beta(a)\beta(b)z$, we get the following Calabi–Yau operators

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $b$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{6}$ | $\frac{5}{6}$ |
| Nr. | 3* | – | 6* | 14* | 4* | 4** | 8* | 8** |
| $a$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |
| $b$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{6}$ | $\frac{5}{6}$ | $\frac{3}{8}$ | $\frac{5}{8}$ | $\frac{5}{12}$ | $\frac{7}{12}$ |
| Nr. | 10* | 10** | 13* | 13** | 7* | 7** | 9* | 9** |

The case $a = \frac{1}{2}$ and $b = \frac{1}{3}$ is not listed here, since the corresponding operator is $\mathrm{Sym}^3$ of a second order operator.

(iv) The $P_2(4,6,8)$ case:
Having performed the transformation $z \mapsto \beta(a)^2\beta(b)^2 z$, we get the following Calabi–Yau operators

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $b$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ |
| Nr. | 111 | 110 | 30 | 112 | 141 | 142 | 196 | 143 |
| $a$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $b$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ |
| Nr. | 189 | 194 | 197 | 199 | 190 | 195 | 198 | 61 |

## Acknowledgements

We thank Duco van Straten for various fruitful discussions and suggestions concerning the content of this article. We also thank Michael Dettweiler, Barbara Jung and the referees for helpful comments and remarks.

## Appendix A. Subgroup structure of $\mathrm{Sp}_4(\mathbb{C})$

We give an overview of the maximal irreducible subgroups in $\mathrm{Sp}_4(\mathbb{C})$ and their behavior under taking the exterior product.

**Lemma A.1.** *The maximal semisimple connected subgroups of* $\mathrm{Sp}_4(\mathbb{C})$ *are contained in one of the following classes*:

1. $(\mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})).2$,

2. $\mathrm{GL}_2(\mathbb{C}).2 \cong \mathrm{Sp}_2(\mathbb{C}) \otimes \mathrm{GO}_2(\mathbb{C})$,
3. $\mathrm{Sym}^3\, \mathrm{SL}_2(\mathbb{C})$,

*where .2 denotes a group extension of degree* 2.

**Proof.** A maximal connected semisimple subgroup $G$ of $\mathrm{Sp}_4(\mathbb{C})$ can be written as a product $G = G_1 \cdots G_r$ of simple groups $G_i$. Hence $G_i$ is either a torus or $\mathrm{Sp}_2(\mathbb{C})$. Since the Lie-rank of $\mathrm{Sp}_4(\mathbb{C})$ is two we get $r \leqslant 2$. This gives the claim, cf. Carter (1985, Chapter 1). □

**Corollary A.2.** *Two classes of the maximal irreducibles subgroups in* $\mathrm{Sp}_4(\mathbb{C})$ *become reducible in* $\mathrm{SO}_5(\mathbb{C})$ *taking their antisymmetric square.*

$$\Lambda^2\big(\mathrm{Sp}_2(\mathbb{C}) \otimes \mathrm{GO}_2(\mathbb{C})\big) = \big\langle (A, B) \in \mathrm{GO}_3(\mathbb{C}) \times \mathrm{GO}_2(\mathbb{C}) \;\big|\; \det(A)\det(B) = 1 \big\rangle,$$
$$\Lambda^2\big(\mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})\big).2 = \mathrm{GO}_4(\mathbb{C}),$$

*where* $\mathrm{GO}_4(\mathbb{C})$ *is naturally embedded into* $\mathrm{SO}_5(\mathbb{C})$.

**Proof.** The claims follow from the identities

$$\Lambda^2(V_1 \otimes V_2) = \Lambda^2 V_1 \otimes \mathrm{Sym}^2 V_2 \oplus \mathrm{Sym}^2 V_1 \otimes \Lambda^2 V_2,$$
$$\Lambda^2(V_1 \oplus V_2) = \Lambda^2(V_1) \oplus V_1 \otimes V_2 \oplus \Lambda^2 V_2. \qquad \square$$

**Corollary A.3.** *Let $H$ be an irreducible proper subgroup of* $\mathrm{Sp}_4(\mathbb{C})$. *Then the following hold* (*up to conjugation of $H$*).

1. *If $H$ contains a unipotent element with Jordan form* $\mathbf{J}(4)$ *then* $H \subseteq \mathrm{Sym}^3\, \mathrm{Sp}_2(\mathbb{C})$.
2. *If $H$ contains a transvection then* $H \subseteq (\mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})).2$.
3. *If all non-trivial unipotent elements in $H$ have the Jordan form* $(\mathbf{J}(2), \mathbf{J}(2))$ *then* $H \subseteq \mathrm{SL}_2 \otimes \mathrm{GO}_2(\mathbb{C})$.

**Proof.** The claims follow from Lemma A.1. □

## References

Almkvist, G., 2006. Calabi–Yau differential equations of degree 2 and 3 and Yifan Yang's pullback. Preprint, http://arxiv.org/abs/math/0612215.

Almkvist, G., van Enckevort, C., van Straten, D., Zudilin, W., 2010. Tables of Calabi–Yau equations. Preprint, http://arxiv.org/abs/math/0507430.

Almkvist, G., Zudilin, W., 2005. Differential equations, mirror maps and zeta values. In: Mirror Symmetry V. In: Stud. Adv. Math., vol. 38. American Mathematical Society, pp. 481–515.

André, Y., 1989. G-Functions and Geometry. Aspects Math., vol. E13. Vieweg und Sohn.

Bailey, W.N., 1935. Generalized Hypergeometric Series. Cambridge University Press.

Baldassarri, F., Dwork, B., 1979. On second order linear differential equations with algebraic solutions. Amer. J. Math. 101, 42–76.

Beukers, F., Heckman, G., 1989. Monodromy for the hypergeometric function ${}_nF_{n-1}$. Invent. Math. 95 (2), 325–354.

Candelas, P., de la Ossa, X.C., Green, P.S., Parkes, L., 1998. A pair of Calabi–Yau manifolds as an exactly soluble superconformal field theory. In: Mirror Symmetry I. In: Stud. Adv. Math., vol. 9. American Mathematical Society, pp. 31–95.

Carter, R.W., 1985. Finite Groups of Lie Type. Conjugacy Classes and Complex Characters. Pure Appl. Math. John Wiley & Sons, Inc.

Deligne, P., 1970. Équations Différentielles à Points Singuliers Réguliers. Lecture Notes in Math., vol. 163. Springer-Verlag, Heidelberg.

Dettweiler, M., Reiter, S., 2000. An algorithm of Katz and its application to the inverse Galois problem. J. Symbolic Comput. 30 (6), 761–798.

Dettweiler, M., Reiter, S., 2007. Middle convolution of Fuchsian systems and the construction of rigid differential systems. J. Algebra 318 (1), 1–24.

Dwork, B., Gerotto, G., Sullivan, F.J., 1994. An Introduction to G-Functions. Ann. of Math. Stud., vol. 133. Princeton University Press.

van Enckevort, C., van Straten, D., 2004. Monodromy calculations of fourth order equations of Calabi–Yau type. In: Mirror Symmetry V. In: Stud. Adv. Math., vol. 38. American Mathematical Society, pp. 539–559.

Garbagnati, A., van Geemen, B., 2010. Examples of Calabi–Yau threefolds parametrised by Shimura varieties. Rend. Semin. Mat. Univ. Politec. Torino 68, 65–81.

Ince, E.L., 1956. Ordinary Differential Equations. Dover, London.

Iwasaki, K., Kimura, H., Shimomura, S., Yoshida, M., 1991. From Gauss to Painlevé—A Modern Theory of Special Functions. Aspects Math. Vieweg, Braunschweig.

Katz, N.M., 1996. Rigid Local Systems. Ann. of Math. Stud., vol. 139. Princeton University Press.

Kontsevich, M., Zagier, D.B., 2001. Periods. In: Mathematics Unlimited—2001 and Beyond. Springer, Berlin, pp. 771–808.

Krattenthaler, C., Rivoal, T., 2010. On the integrality of Taylor coefficients of mirror maps. Duke Math. J. 151, 175–218.

Krattenthaler, C., Rivoal, T., 2011. Multivariate p-adic formal congruences and integrality of Taylor coefficients of mirror maps. Sémin. Congr. 23, 301–329.

Levelt, A.H., 1961. Hypergeometric functions. Thesis, University of Amsterdam.

van der Put, M., Singer, M.F., 2002. Galois Theory of Linear Differential Equations, second ed. Grundlehren Math. Wiss., vol. 328. Springer-Verlag, Berlin.

Riemann, B., 1857. Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen 7.

Scott, L.L., 1977. Matrices and cohomology. Ann. Math., 473–492.

Simpson, C., 1990. Transcendental aspects of the Riemann–Hilbert correspondence. Illinois J. Math. 34 (2), 368–391.

Simpson, C., 1992. Higgs bundles and local systems. Publ. Math. IHES 75, 5–95.

Singer, M.F., 1996. Testing reducibility of linear differential operators: a group theoretic perspective. Appl. Algebra Engrg. Comm. Comput. 7 (2), 77–104.

Strambach, K., Völklein, H., 1999. On linearly rigid tuples. J. Reine Angew. Math. 510, 57–62.